CONFERENCIA CLASE
MÉTODOS DE INTEGRACIÓN Y APLICACIONES
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INTEGRACIÓN POR PARTES

$$u=f(x) \quad y \quad v=g(x) \quad ; \quad d(uv)=udv+vdu$$

$$udv=d(uv)-vdu$$

$$\int udv=uv-\int vdu$$

$$\int x\, sen2x\, dx$$

$$u=x \quad \Rightarrow \quad du=dx \quad ; \quad dv=sen2x\, dx \quad \Rightarrow \quad v=-\frac{\cos 2x}{2}$$

$$\int udv=uv-\int vdu$$

$$\int x\, sen2x\, dx=-\frac{x\cos 2x}{2}-\int\left(-\frac{\cos 2x}{2}\right)dx$$

$$\int x\, sen2x\, dx=-\frac{x\cos 2x}{2}+\frac{1}{2}\int \cos 2x\, dx$$

$$\therefore \quad \int x\, sen2x\, dx=-\frac{x\cos 2x}{2}+\frac{sen2x}{4}+C$$

$$\int \ln x\, dx$$

$$u=\ln x \quad \Rightarrow \quad du=\frac{dx}{x} \quad ; \quad dv=dx \quad \Rightarrow \quad v=x$$

$$\int \ln x\, dx=x\ln x-\int x\,\frac{dx}{x}$$

$$\int \ln x\, dx=x\ln x-\int dx$$

$$\therefore \quad \int \ln x\, dx=x\ln x-x+C$$

$$\int x^2e^{3x}dx$$

$$u = x^2 \quad \Rightarrow \quad du = 2x\,dx \quad ; \quad dv = e^{3x}dx \quad \Rightarrow \quad v = \frac{e^{3x}}{3}$$

$$\int x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3} - \int \frac{e^{3x}}{3}(2x)\,dx$$

$$\int x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\int x e^{3x} dx$$

$$\int x e^{3x} dx$$

$$u = x \quad \Rightarrow \quad du = dx \quad ; \quad dv = e^{3x}dx \quad \Rightarrow \quad v = \frac{e^{3x}}{3}$$

$$\int x e^{3x} dx = \frac{x e^{3x}}{3} - \int \frac{e^{3x}}{3}\,dx \quad \Rightarrow \quad \int x e^{3x} dx = \frac{x e^{3x}}{3} - \frac{e^{3x}}{9} + C_1$$

$$\int x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3} - \frac{2}{3}\left(\frac{x e^{3x}}{3} - \frac{e^{3x}}{9}\right) + C$$

$$\int x^2 e^{3x} dx = \frac{x^2 e^{3x}}{3} - \frac{2x e^{3x}}{9} + \frac{2e^{3x}}{27} + C$$

$$\boxed{\int x \;\ln\left(1 + \frac{1}{x}\right)\; dx}$$

$$\int x \;\ln\left(1 + \frac{1}{x}\right)\; dx = \int x \ln\frac{x+1}{x}\; dx$$

$$= \int x\;\left[\ln(x+1) - \ln x\right]\; dx$$

$$= \int x \ln(x+1)\; dx - \int x \ln x\; dx$$

$$\int x \ln(x+1)\,dx$$

$$u = \ln(x+1) \Rightarrow du = \frac{dx}{x+1}\; ; \; dv = x\; dx \Rightarrow v = \frac{x^2}{2}$$

$$\therefore \int x\ln(x+1)\ dx=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\frac{x^2}{x+1}\ dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{1}{2}\int\left(x-1+\frac{1}{x+1}\right)\ dx$$

$$=\frac{x^2}{2}\ln(x+1)-\frac{x^2}{4}+\frac{x}{2}-\frac{1}{2}\ln(x+1)+C$$

$$=\frac{x^2-1}{2}\ln(x+1)-\frac{x^2}{4}+\frac{x}{2}+C$$

$$\int x\ln x dx$$

$$u=\ln x\ \Rightarrow\ du=\frac{dx}{x}\ ;\ dv=x\ \ dx\ \Rightarrow\ v=\frac{x^2}{2}$$

$$=\frac{x^2}{2}\ln x-\frac{1}{2}\int x\ \ dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$$

$$\therefore \quad \int x\ \ln\left(1+\frac{1}{x}\right)dx=\frac{x^2-1}{2}\ln(x+1)-\frac{x^2}{2}\ln x+\frac{x}{2}+C$$

$$\int sen x \ln\tan x dx$$

$$u=\ln\tan x\Rightarrow\ du=\frac{\sec^2 x}{\tan x}dx=\frac{dx}{sen\ \ x\cos x}$$

$$dv=sen\ \ x\ \ dx\Rightarrow v=-\cos x$$

$$\therefore \int sen x\ \ln\tan x\ \ dx$$

$$=-\cos x\ \ln\tan x-\int(-\cos x)\frac{dx}{sen x\ \ \cos x}$$

$$=-\cos x\ \ln\tan x+\int\csc x\ \ dx$$

$$=-\cos x\ \ln\tan x+\ln(\csc x-\cot x)+C$$

$$\int \frac{x\cos x}{sen^3 x}dx$$

$$u = x \Rightarrow du = dx \ ; \ dv = \frac{\cos x}{sen^3 x}dx$$

$$w = senx \Rightarrow dw = \cos x dx$$

$$\Rightarrow v = \int \frac{dw}{w^3} = -\frac{1}{2w^2} = -\frac{1}{2sen^2 x}$$

$$\therefore \int \frac{x \ \cos x}{sen^3 x}dx = -\frac{x}{2sen^2 x} + \frac{1}{2}\int \frac{1}{sen^2 x}dx$$

$$= -\frac{x}{2sen^2 x} + \frac{1}{2}\int \csc^2 x\, dx = -\frac{x}{2sen^2 x} - \frac{\cot x}{2} + C$$

DIFERENCIALES TRIGONOMÉTRICAS

$$\int sen^2 3x \cos^2 3x dx$$

$$\int \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right)\left(\frac{1}{2} + \frac{1}{2}\cos 6x\right)dx =$$

$$\int \left(\frac{1}{4} + \frac{1}{4}\cos 6x - \frac{1}{4}\cos 6x - \frac{1}{4}\cos^2 6x\right)dx =$$

$$\int \left(\frac{1}{4} - \frac{1}{4}\cos^2 6x\right)dx = \frac{1}{4}\int dx - \frac{1}{4}\int \cos^2 6x\, dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\int \left(\frac{1}{2} + \frac{1}{2}\cos 12x\right)dx =$$

$$= \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\int \cos 12x\, dx = \frac{1}{8}\int dx - \frac{1}{8}\int \cos 12x\, dx$$

$$\therefore \quad \int sen^2 3x \cos^2 3x dx = \frac{x}{8} - \frac{sen12x}{96} + C$$

$$\int \frac{\cos^5 x}{\sqrt{senx}}dx$$

$$= \int \cos^4 x sen^{-\frac{1}{2}} x \cos x \, dx = \int \left(1 - sen^2 x\right)^2 sen^{-\frac{1}{2}} x \cos x \, dx =$$

$$= \int \left(1 - 2sen^2 x + sen^4 x\right) sen^{-\frac{1}{2}} x \cos x \, dx =$$

$$= \int sen^{-\frac{1}{2}} x \cos x \, dx - 2\int sen^{\frac{3}{2}} x \cos x \, dx + \int sen^{\frac{7}{2}} x \cos x \, dx$$

$$u = senx \quad \Rightarrow \quad du = \cos x \, dx$$

$$\int u^{-\frac{1}{2}} du - 2\int u^{\frac{3}{2}} du + \int u^{\frac{7}{2}} du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{2u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{9}{2}}}{\frac{9}{2}} + C$$

$$\therefore \quad \int \frac{\cos^5 x}{\sqrt{senx}} dx = 2sen^{\frac{1}{2}} x - \frac{4}{5} sen^{\frac{5}{2}} x + \frac{2}{9} sen^{\frac{9}{2}} x + C$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^6 x \, dx$$

$$\int \sec^6 x \, dx = \int \left(\tan^2 x + 1\right)^2 \sec^2 x \, dx =$$

$$= \int \left(\tan^4 x + 2\tan^2 x + 1\right) \sec^2 x \, dx$$

$$= \int \left(\tan^4 x \sec^2 x + 2\tan^2 x \sec^2 x + \sec^2 x\right) dx$$

$$u = \tan x \quad \Rightarrow \quad du = \sec^2 x \, dx$$

$$\int u^4 du + 2\int u^2 du + \int du = \frac{u^5}{5} + \frac{2u^3}{3} + u + C$$

$$= \frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x + C$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^6 x \, dx = \left[\frac{\tan^5 x}{5} + \frac{2\tan^3 x}{3} + \tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} =$$

$$=\frac{\tan^5\frac{\pi}{4}}{5}+\frac{2\tan^3\frac{\pi}{4}}{3}+\tan\frac{\pi}{4}-\frac{\tan^5\frac{\pi}{6}}{5}-\frac{2\tan^3\frac{\pi}{6}}{3}-\tan\frac{\pi}{6}$$

$$=\frac{1}{5}+\frac{2}{3}+1-\frac{\left(\frac{1}{\sqrt{3}}\right)^5}{5}-\frac{2\left(\frac{1}{\sqrt{3}}\right)^3}{3}-\frac{1}{\sqrt{3}}$$

$$=\frac{28}{15}-\frac{1}{45\sqrt{3}}-\frac{2}{9\sqrt{3}}-\frac{1}{\sqrt{3}}=\frac{28}{15}-\frac{56}{45\sqrt{3}}$$

$$\therefore\quad \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^6 x\,dx\approx 1.1482$$

$$\int\tan^3 4x\,dx$$

$$\int\tan^3 4x\,dx=\int\tan^2 4x\tan 4x\,dx=\int\left(\sec^2 4x-1\right)\tan 4x\,dx$$

$$=\int\tan 4x\sec^2 4x\,dx-\int\tan 4x\,dx$$

Para la primera:

$$u=\tan 4x\quad\Rightarrow\quad du=4\sec^2 4x\,dx$$

$$\frac{1}{4}\int u\,du=\frac{1}{4}\frac{u^2}{2}+C_1=\frac{\tan^2 4x}{8}+C_1$$

La segunda es directa:

$$\int\tan 4x\,dx=\frac{1}{4}\ln\left|\sec 4x\right|+C_2$$

$$\therefore\quad\int\tan^3 4x\,dx=\frac{\tan^2 4x}{8}+\frac{1}{4}\ln\left|\sec 4x\right|+C$$

La primera se puede resolver también como:

$$\int\tan 4x\sec^2 4x\,dx=\int\sec 4x\sec 4x\tan 4x\,dx$$

$$v=\sec 4x\quad\Rightarrow\quad dv=4\sec 4x\tan 4x\,dx$$

$$\frac{1}{4}\int u\,du=\frac{1}{4}\frac{u^2}{2}+C_3\quad\Rightarrow\quad\int\tan 4x\sec^2 4x\,dx=\frac{\sec^2 4x}{8}+C_3$$

$$\therefore \quad \int \tan^3 4x\, dx = \frac{\sec^2 4x}{8} + \frac{1}{4}\ln\left|\sec 4x\right| + C$$

$$\int \cot^6 x\, dx$$

$$\int \cot^6 x\, dx = \int \cot^4 x \cot^2 x\, dx = \int \cot^4 x\left(\csc^2 x - 1\right) dx$$

$$\int \left(\cot^4 x \csc^2 x - \cot^4 x\right) dx$$

$$= \int \cot^4 x \csc^2 x\, dx - \int \cot^4 x\, dx$$

$$= \int \cot^4 x \csc^2 x\, dx - \int \cot^2 x\left(\csc^2 x - 1\right) dx$$

$$= \int \cot^4 x \csc^2 x\, dx - \int \left(\cot^2 x \csc^2 x - \cot^2 x\right) dx$$

$$= \int \cot^4 x \csc^2 x\, dx - \int \cot^2 x \csc^2 x\, dx + \int \left(\csc^2 x - 1\right) dx =$$

$$= \int \cot^4 x \csc^2 x\, dx - \int \cot^2 x \csc^2 x\, dx + \int \csc^2 x\, dx - \int dx$$

$$u = \cot x \quad \Rightarrow \quad du = -\csc^2 x\, dx$$

$$-\int u^4 du + \int u^2 du - \int du - \int dx = -\frac{u^5}{5} + \frac{u^3}{3} - u - x + C_1$$

$$\therefore \quad \int \cot^6 x\, dx = -\frac{\cot^5 x}{5} + \frac{\cot^3 x}{3} - \cot x - x + C_1$$

$$\int \sec^3 x \tan^5 x\, dx$$

$$\int \sec^3 x \tan^5 x\, dx = \int \sec^2 x \tan^4 x \sec x \tan x\, dx$$

$$= \int \sec^2 x\left(\sec^2 x - 1\right)^2 \sec x \tan x\, dx$$

$$= \int \sec^2 x\left(\sec^4 x - 2\sec^2 x + 1\right) \sec x \tan x\, dx$$

$$= \int \sec^6 x \sec x \tan x\, dx - 2\int \sec^4 x \sec x \tan x\, dx$$

$$+ \int \sec^2 x \sec x \tan x\, dx$$

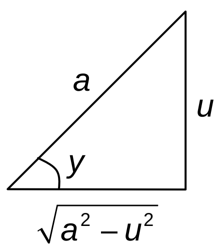
$$u = \sec x \quad \Rightarrow \quad du = \sec x \tan x\, dx$$

$$\int u^6 du - 2\int u^4 du + \int u^2 du = \frac{u^7}{7} - \frac{2u^5}{5} + \frac{u^3}{3} + C$$

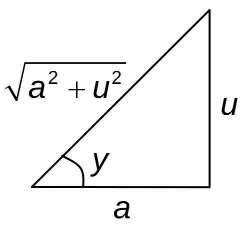
$$\int \sec^3 x \tan^5 x\, dx = \frac{\sec^7 x}{7} - \frac{2\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

SUSTITUCIÓN TRIGONOMÉTRICA

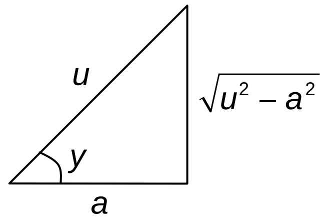
i) $\left(a^2 - u^2\right)^{\frac{1}{2}}$



ii) $\left(a^2 + u^2\right)^{\frac{1}{2}}$



iii) $\left(u^2 - a^2\right)^{\frac{1}{2}}$

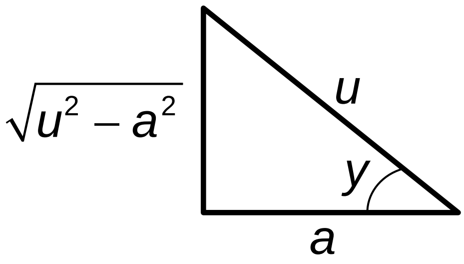


$$\int \sqrt{9x^2-16}\ dx$$

$$u^2=9x^2 \quad ; \quad u=3x \quad ; \quad du=3dx$$

$$a^2=16 \quad ; \quad a=4$$

$$\Rightarrow \quad \frac{1}{3}\int \sqrt{u^2-a^2}\ du$$



$$u=a\sec y$$

$$du=a\sec y\tan y\ dy$$

$$\sqrt{u^2-a^2}=a\tan y$$

$$\Rightarrow \quad \frac{1}{3}\int a\tan y\ a\sec y\tan y\ dy=\frac{a^2}{3}\int \sec y\tan^2 y\ dy$$

$$\frac{a^2}{3}\int \sec y\left(\sec^2 y-1\right)dy$$

$$=\frac{a^2}{3}\int \sec^3 y\ dy-\frac{a^2}{3}\int \sec y\ dy$$

La primera se resuelve por partes y:

$$\int \sec^3 y\ dy=\int \sec y\sec^2 y\ dy$$

$$u=\sec y \quad ; \quad du=\sec y\tan y dy$$

$$dv=\sec^2 y dy \quad ; \quad v=\tan y$$

$$\int \sec^3 y\ dy=\sec y\tan y-\int \sec y\tan^2 y\ dy$$

$$\int \sec^3 y\,dy = \sec y \tan y - \int \sec y\left(\sec^2 y - 1\right)dy$$

$$\int \sec^3 y\,dy = \sec y \tan y - \int \sec^3 y\,dy + \int \sec y\,dy$$

$$2\int \sec^3 y\,dy = \sec y \tan y + \int \sec y\,dy$$

$$\therefore \quad \int \sec^3 y\,dy = \frac{1}{2}\sec y \tan y + \frac{1}{2}\ln\left|\sec y + \tan y\right| + C$$

Luego:

$$= \frac{a^2}{3}\left(\frac{1}{2}\sec y \tan y + \frac{1}{2}\ln\left|\sec y + \tan y\right|\right)$$

$$- \frac{a^2}{3}\ln\left|\sec y + \tan y\right| + C$$

$$= \frac{a^2}{6}\sec y \tan y - \frac{a^2}{6}\ln\left|\sec y + \tan y\right| + C$$

Se sustituyen las funciones trigonométricas y:

$$\frac{a^2}{6}\frac{u}{a}\frac{\sqrt{u^2 - a^2}}{a} - \frac{a^2}{6}\ln\left|\frac{u}{a} + \frac{\sqrt{u^2 - a^2}}{a}\right| + C$$

$$= \frac{u\sqrt{u^2 - a^2}}{6} - \frac{a^2}{6}\ln\left|\frac{u + \sqrt{u^2 - a^2}}{a}\right| + C$$

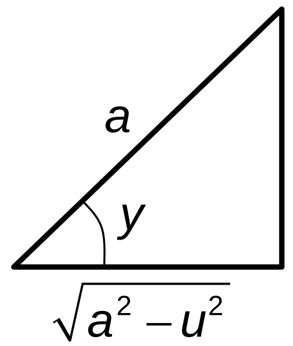
$$\therefore \quad \int \sqrt{9x^2 - 16}\,dx = \frac{3x\sqrt{9x^2 - 16}}{6} - \frac{16}{6}\ln\left|\frac{3x + \sqrt{9x^2 - 16}}{4}\right| + C$$

$$= \frac{x\sqrt{9x^2 - 16}}{2} - \frac{8}{3}\ln\left|\frac{3x + \sqrt{9x^2 - 16}}{4}\right| + C$$

$$\int \frac{x^2 dx}{\sqrt{1 - 4x^2}}$$

$$u^2 = 4x^2 \quad \Rightarrow \quad u = 2x \quad \Rightarrow \quad du = 2dx \quad ; \quad a^2 = 1 \quad \Rightarrow \quad a = 1$$

$$\int \frac{x^2 dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{\frac{u^2}{4}du}{\sqrt{a^2-u^2}} = \frac{1}{8}\int \frac{u^2 du}{\sqrt{a^2-u^2}}$$



$$u = a sen y \quad \Rightarrow \quad du = a\cos y\, dy$$

$$u^2 = a^2 sen^2 y \quad ; \quad \sqrt{a^2-u^2} = a\cos y$$

$$\frac{1}{8}\int \frac{u^2 du}{\sqrt{a^2-u^2}} = \frac{1}{8}\int \frac{a^2 sen^2 y\, a\cos y dy}{a\cos y}$$

$$= \frac{a^2}{8}\int sen^2 y\, dy = \frac{a^2}{8}\int\left(\frac{1}{2}-\frac{1}{2}\cos 2y\right)dy$$

$$= \frac{a^2}{16}\int dy - \frac{a^2}{16}\int \cos 2y\, dy = \frac{a^2}{16}y - \frac{a^2}{32}sen 2y + C$$

$$\frac{a^2}{16} angsen\frac{u}{a} - \frac{a^2}{32}2 sen y\cos y + C$$

$$= \frac{a^2}{16} angsen\frac{u}{a} - \frac{a^2}{16} sen y\cos y + C$$

$$= \frac{a^2}{16} angsen\frac{u}{a} - \frac{a^2}{16}\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a} + C$$

$$= \frac{a^2}{16} angsen\frac{u}{a} - \frac{u\sqrt{a^2-u^2}}{16} + C$$

$$= \frac{1}{16} angsen\frac{2x}{1} - \frac{2x\sqrt{1-4x^2}}{16} + C$$

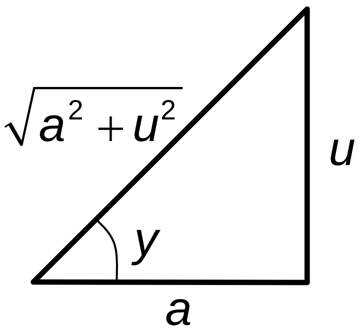
$$= \frac{1}{16} angsen 2x - \frac{x\sqrt{1-4x^2}}{8} + C$$

$$\therefore \quad \int \frac{x^2 dx}{\sqrt{1-4x^2}} = \frac{1}{16} angsen 2x - \frac{x\sqrt{1-4x^2}}{8} + C$$

$$\int \frac{dx}{x\sqrt{64x^2+25}}$$

$$u^2 = 64x^2 \quad \Rightarrow \quad u = 8x \quad \Rightarrow \quad du = 8\ dx \quad ; \quad a^2 = 25 \quad \Rightarrow \quad a = 5$$

$$\int \frac{dx}{x\sqrt{64x^2+25}} = \frac{1}{8}\int \frac{du}{\frac{u}{8}\sqrt{64x^2+25}} = \int \frac{du}{u\sqrt{u^2+a^2}}$$



$$\int \frac{du}{u\sqrt{u^2+a^2}}$$

$$u = a\tan y \quad \Rightarrow \quad du = a\sec^2 y\ dy \quad ; \quad \sqrt{a^2+u^2} = a\sec y$$

$$\int \frac{du}{u\sqrt{u^2+a^2}} = \int \frac{a\sec^2 y\ dy}{a\tan y\ a\sec y}$$

$$= \frac{1}{a}\int \frac{\sec y\ dy}{\tan y} = \frac{1}{a}\int \frac{\frac{1}{\cos y}}{\frac{sen y}{\cos y}} dy = \frac{1}{a}\int \csc y\ dy$$

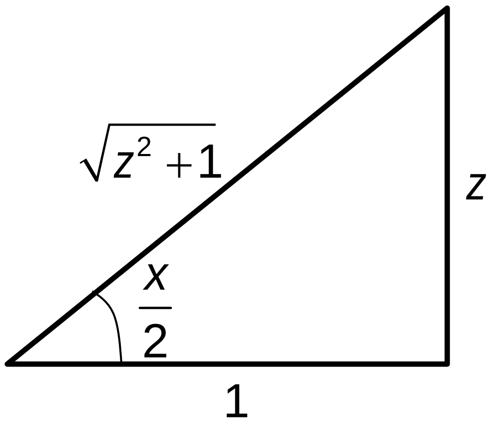
$$= \frac{1}{a}\ln\left|\csc y - \cot y\right| + C$$

$$\frac{1}{a}\ln\left|\csc y-\cot y\right|+C=\frac{1}{a}\ln\left|\frac{\sqrt{u^2+a^2}}{u}-\frac{a}{u}\right|+C$$

$$=\frac{1}{5}\ln\left|\frac{\sqrt{64x^2+25}-5}{8x}\right|+C$$

$$\therefore\quad \int\frac{dx}{x\sqrt{64x^2+25}}=\frac{1}{5}\ln\left|\frac{\sqrt{64x^2+25}-5}{8x}\right|+C$$

INTEGRACIÓN POR SUSTITUCIÓN TRIGONOMÉTRICA DEL ANGULO MEDIO



$$z=\tan\frac{x}{2}$$

$$sen x=sen 2\left(\frac{x}{2}\right)=2sen\frac{x}{2}\cos\frac{x}{2}=2\frac{z}{\sqrt{z^2+1}}\frac{1}{\sqrt{z^2+1}}=\frac{2z}{z^2+1}$$

$$\cos x=\cos 2\left(\frac{x}{2}\right)=\cos^2\frac{x}{2}-sen^2\frac{x}{2}=\frac{1}{z^2+1}-\frac{z^2}{z^2+1}=\frac{1-z^2}{z^2+1}$$

$$\frac{x}{2}=ang\tan z\quad\Rightarrow\quad x=2ang\tan z\quad\Rightarrow\quad dx=\frac{2dz}{z^2+1}$$

$$\int\frac{2\,dx}{sen x+\cos x}$$

$$\int \frac{2\,dx}{\operatorname{sen} x + \cos x} = \int \frac{2\frac{2\,dz}{z^2+1}}{\frac{2z}{z^2+1} + \frac{1-z^2}{z^2+1}}$$

$$= 4\int \frac{dz}{2z+1-z^2} = 4\int \frac{dz}{-\left(z^2-2z-1\right)}$$

$$= 4\int \frac{dz}{-\left(z^2-2z+1-1-1\right)} = 4\int \frac{dz}{2-\left(z-1\right)^2}$$

$$u^2 = \left(z-1\right)^2 \quad \Rightarrow \quad u = z-1$$

$$\Rightarrow \quad du = dz \quad ; \quad a^2 = 2 \quad \Rightarrow \quad a = \sqrt{2}$$

$$4\int \frac{dz}{2-\left(z-1\right)^2} = 4\int \frac{du}{a^2-u^2} =$$

$$4\frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C = \frac{2}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+z-1}{\sqrt{2}-z+1}\right| + C$$

$$\therefore \quad \int \frac{2\,dx}{\operatorname{sen} x + \cos x} = \sqrt{2}\ln\left|\frac{\sqrt{2}-1+\tan\frac{x}{2}}{\sqrt{2}+1-\tan\frac{x}{2}}\right| + C$$

DESCOMPOSICIÓN EN FRACCIONES RACIONALES

$$i) \quad \left(ax+b\right)^n \quad ; \quad n \geq 1$$

$$\frac{A_1}{ax+b} + \frac{A_2}{\left(ax+b\right)^2} + \cdots + \frac{A_n}{\left(ax+b\right)^n}$$

$$A_i \quad ; \quad i = 1, 2, \ldots, n \in \mathbb{R}$$

$$ii) \quad \left(ax^2+bx+c\right)^n \quad ; \quad n \geq 1 \quad y \quad b^2-4ac<0$$

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\cdots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)^n}$$

$$A_i \quad y \quad B_i \quad ; \quad i=1,2,\ldots n\in\mathbb{R}$$

$$\int\frac{x-6}{x^2-3x-10}\,dx$$

$$x^2-3x-10=\left(x+2\right)\left(x-5\right)$$

$$\frac{x-6}{x^2-3x-10}=\frac{A_1}{x+2}+\frac{A_2}{x-5}$$

$$\frac{x-6}{x^2-3x-10}=\frac{A_1\left(x-5\right)+A_2\left(x+2\right)}{\left(x+2\right)\left(x-5\right)}$$

$$\Rightarrow \quad \frac{x-6}{x^2-3x-10}=\frac{A_1x-5A_1+A_2x+2A_2}{\left(x+2\right)\left(x-5\right)}$$

$$x-6=A_1x-5A_1+A_2x+2A_2$$

$$\begin{matrix} 1=A_1+A_2 \\ -6=-5A_1+2A_2 \end{matrix} \quad\Rightarrow\quad \begin{cases} A_1+A_2=1 \\ -5A_1+2A_2=-6 \end{cases}$$

$$A_1=1-A_2 \quad ; \quad -5\left(1-A_2\right)+2A_2=-6$$

$$\Rightarrow \quad -5+5A_2+2A_2=-6$$

$$7A_2=-1 \quad\Rightarrow\quad A_2=-\frac{1}{7} \quad ; \quad A_1=1-\left(-\frac{1}{7}\right) \quad\Rightarrow\quad A_1=\frac{8}{7}$$

$$\int\frac{x-6}{x^2-3x-10}\,dx=\int\frac{\frac{8}{7}}{x+2}\,dx+\int\frac{-\frac{1}{7}}{x-5}\,dx$$

$$\int\frac{x-6}{x^2-3x-10}\,dx=\frac{8}{7}\int\frac{dx}{x+2}-\frac{1}{7}\int\frac{dx}{x-5}$$

$$u=x+2 \quad\Rightarrow\quad du=dx \qquad ; \qquad v=x-5 \quad\Rightarrow\quad dv=dx$$

$$\frac{8}{7}\int\frac{dx}{x+2}=\frac{8}{7}\int\frac{du}{u}=\frac{8}{7}\ln\left|u\right|+C_1=\frac{8}{7}\ln\left|x+2\right|+C_1$$

$$-\frac{1}{7}\int\frac{dx}{x-5}=-\frac{1}{7}\int\frac{dv}{v}=-\frac{1}{7}\ln|v|+C_2=-\frac{1}{7}\ln|x-5|+C_2$$

$$\therefore\quad \int\frac{x-6}{x^2-3x-10}\,dx=\frac{8}{7}\ln|x+2|-\frac{1}{7}\ln|x-5|+C=\frac{1}{7}\ln\left|\frac{(x+2)^8}{x-5}\right|+C$$

$$\int\frac{x^3-2x^2+x-1}{x^4-1}dx$$

$$\frac{x^3-2x^2+x-1}{x^4-1}=\frac{x^3-2x^2+x-1}{(x^2-1)(x^2+1)}=\frac{x^3-2x^2+x-1}{(x-1)(x+1)(x^2+1)}$$

$$\frac{x^3-2x^2+x-1}{x^4-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

$$x^3-2x^2+x-1=$$

$$A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$$

$$x^3-2x^2+x-1=$$

$$A(x^3+x+x^2+1)+B(x^3+x-x^2-1)+Cx^3-Cx+Dx^2-D$$

$$x^3-2x^2+x-1=$$

$$Ax^3+Ax+Ax^2+A+Bx^3+Bx-Bx^2-B+Cx^3-Cx+Dx^2-D$$

$$\begin{cases}1=A+B+C\\-2=A-B+D\\1=A+B-C\\-1=A-B-D\end{cases}\Rightarrow\begin{cases}A+B+C=1\\A-B+D=-2\\A+B-C=1\quad\Rightarrow\quad C=A+B-1\\A-B-D=-1\quad\Rightarrow\quad D=A-B+1\end{cases}$$

$$\begin{cases}A+B+A+B-1=1\\A-B+A-B+1=-2\end{cases}$$

$$\Rightarrow\quad\begin{cases}2A+2B=2\\2A-2B=-3\end{cases}\Rightarrow\quad 4A=-1\quad\Rightarrow\quad \begin{array}{l}A=-\frac{1}{4}\\B=\frac{5}{4}\end{array}$$

$$C=-\frac{1}{4}+\frac{5}{4}-1 \quad \Rightarrow \quad C=0 \quad ; \quad D=-\frac{1}{4}-\frac{5}{4}+1 \quad \Rightarrow \quad D=-\frac{1}{2}$$

$$\int \frac{x^3-2x^2+x-1}{x^4-1}dx=\int \frac{-\frac{1}{4}}{x-1}dx+\int \frac{\frac{5}{4}}{x+1}+\int \frac{0x+\left(-\frac{1}{2}\right)}{x^2+1}dx$$

$$\int \frac{x^3-2x^2+x-1}{x^4-1}dx=-\frac{1}{4}\ln|x-1|+\frac{5}{4}\ln|x+1|-\frac{1}{2}ang\tan x+C$$

$$\therefore \quad \int \frac{x^3-2x^2+x-1}{x^4-1}dx=\frac{1}{4}\ln\left|\frac{(x+1)^5}{x-1}\right|-\frac{1}{2}ang\tan x+C$$

$$\int \frac{2x+1}{x^3-8}\; dx$$

$$\frac{2x+1}{x^3-8}=\frac{2x+1}{(x-2)\left(x^2+2x+4\right)}$$

$$=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

$$2x+1=A\left(x^2+2x+4\right)+(Bx+C)(x-2)$$

$$2x+1=Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$$

$$0=A+B$$

$$2=2A-2B+C$$

$$1=4A-2C$$

$$B=-A \Rightarrow \begin{matrix} 2=4A+C \\ 1=4A-2C \end{matrix}$$

$$\Rightarrow \begin{matrix} 2=4A+C \\ -1=-4A+2C \end{matrix} \Rightarrow 1=3C \Rightarrow \begin{matrix} C=\dfrac{1}{3} \\ A=\dfrac{5}{12} \\ B=-\dfrac{5}{12} \end{matrix}$$

$$\int \frac{2x+1}{x^3-8}\;dx=\int \frac{\frac{5}{12}}{x-2}\;dx+\int \frac{-\frac{5}{12}x+\frac{1}{3}}{x^2+2x+4}\;dx$$

$$=\frac{5}{12}\int \frac{dx}{x-2}-\frac{1}{12}\int \frac{5x-4}{x^2+2x+4}\;dx$$

$$\frac{5}{12}\int \frac{dx}{x-2}=\frac{5}{12}\ln\left|x-2\right|+C_1$$

$$u=x^2+2x+4 \Rightarrow du=\left(2x+2\right)dx=2\left(x+1\right)dx$$

$$\int \frac{5x-4}{x^2+2x+4}\;dx=\int \frac{5x+5-5-4}{x^2+2x+4}\;dx$$

$$=5\int \frac{x+1}{x^2+2x+4}\;dx-9\int \frac{dx}{x^2+2x+4}$$

$$5\int \frac{x+1}{x^2+2x+4}\;dx=\frac{5}{2}\int \frac{du}{u}=\frac{5}{2}\ln\left|u\right|+C_2$$

$$=\frac{5}{2}\ln\left|x^2+2x+4\right|+C_2$$

$$\int \frac{dx}{x^2+2x+4}\;dx=\int \frac{dx}{x^2+2x+1-1+4}$$

$$=\int \frac{dx}{\left(x+1\right)^2+3}$$

$$a^2 = 3 \;;\; a = \sqrt{3}$$

$$u^2 = (x+1)^2 \;;\; u = x+1 \;;\; du = dx$$

$$-9\int \frac{dx}{x^2+2x+4}\; dx = -9\int \frac{dx}{(x+1)^2+3} = -9\int \frac{du}{u^2+a^2}$$

$$= -9 \cdot \frac{1}{a} \, ang\tan\frac{u}{a} + C_3 = -\frac{9}{\sqrt{3}} \, ang\tan\frac{x+1}{\sqrt{3}} + C_3$$

$$\int \frac{2x+1}{x^3-8}\; dx$$

$$= \frac{5}{12}\ln|x-2| - \frac{5}{24}\ln\left|x^2+2x+4\right| + \frac{3}{4\sqrt{3}} \, ang\tan\frac{x+1}{\sqrt{3}} + C$$

$$C = C_1 + C_2 + C_3$$

SUSTITUCIONES DIVERSAS

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$$

$$z^6 = x \quad \Rightarrow \quad x = z^6 \quad \Rightarrow \quad dx = 6z^5\, dz$$

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6z^5\, dz}{\sqrt{z^6}+\sqrt[3]{z^6}} = 6\int \frac{z^5\, dz}{z^3+z^2} = 6\int \frac{z^3\, dz}{z+1}$$

$$6\int \frac{z^3\, dz}{z+1} = 6\int \left(z^2 - z + 1 - \frac{1}{z+1} \right) dz$$

$$= 2z^3 - 3z^2 + 6z - 6\ln|z+1| + C$$

$$\therefore \quad \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - \ln\left(\sqrt[6]{x}+1\right)^6 + C$$

$$\int \frac{dx}{\sqrt{3-x}+\sqrt{(3-x)^3}}$$

$$z^2 = 3 - x \quad \Rightarrow \quad 3 - x = z^2$$

$$\Rightarrow \quad -dx = 2z\,dz \quad \Rightarrow \quad dx = -2z\,dz$$

$$\int \frac{dx}{\sqrt{3-x}+\sqrt{(3-x)^3}} = \int \frac{-2z\,dz}{\sqrt{z^2}+\sqrt{(z^2)^3}} = -2\int \frac{z\,dz}{z+z^3}$$

$$= -2\int \frac{dz}{z^2+1} = -2\,ang\tan z + C$$

$$\therefore \quad \int \frac{dx}{\sqrt{3-x}+\sqrt{(3-x)^3}} = -2\,ang\tan\sqrt{3-x} + C$$

$$i) \quad \int \frac{\sqrt{x^2+9}}{x}\,dx$$

$$\sqrt{x^2+9} = u \quad \Rightarrow \quad x^2+9 = u^2$$

$$\Rightarrow \quad 2x\,dx = 2u\,du \quad \Rightarrow \quad x\,dx = u\,du$$

$$\int \frac{\sqrt{x^2+9}}{x}\,dx = \int \frac{\sqrt{x^2+9}}{x^2}\,xdx = \int \frac{u}{u^2-9}\,u\,du = \int \frac{u^2}{u^2-9}\,du$$

$$\int \frac{u^2}{u^2-9}\,du = \int \left(1+\frac{9}{u^2-9}\right)du = \int du + \int \frac{9}{u^2-9}\,du$$

$$\int \frac{9}{u^2-9}\,du$$

$$v^2 = u^2 \quad \Rightarrow \quad v = u \quad \Rightarrow \quad dv = du$$

$$a^2 = 9 \quad \Rightarrow \quad a = 3$$

$$9\int \frac{dv}{v^2-a^2} = \frac{9}{2a}\ln\left|\frac{v-a}{v+a}\right| + C_1$$

$$= \frac{9}{2(3)}\ln\left|\frac{u-3}{u+3}\right| + C_1 = \frac{3}{2}\ln\left|\frac{u-3}{u+3}\right| + C_1$$

$$\int \frac{u^2}{u^2-9}\,du = u + \frac{3}{2}\ln\left|\frac{u-3}{u+3}\right| + C$$

$$\therefore \quad \int \frac{\sqrt{x^2+9}}{x}dx = \sqrt{x^2+9} + \frac{3}{2}\ln\left|\frac{\sqrt{x^2+9}-3}{\sqrt{x^2+9}+3}\right| + C$$

$$\int \frac{1+x}{1+\sqrt{x}}dx$$

$$\sqrt{x} = u \quad \Rightarrow \quad x = u^2 \Rightarrow \quad dx = 2u\,du$$

$$\int \frac{1+x}{1+\sqrt{x}}dx = \int \frac{1+u^2}{1+u}2u\,du = \int \frac{2u^3+2u}{u+1}du$$

$$\int \frac{2u^3+2u}{u+1}du = \int\left(2u^2-2u+4-\frac{4}{u+1}\right)du$$

$$= 2\int u^2du - 2\int u\,du + 4\int du - 4\int \frac{du}{u+1}$$

$$\int \frac{2u^3+2u}{u+1}du = \frac{2u^3}{3} - u^2 + 4u - 4\ln|u+1| + C$$

$$\therefore \quad \int \frac{1+x}{1+\sqrt{x}}dx = \frac{2x\sqrt{x}}{3} - x + 4\sqrt{x} - 4\ln\left|\sqrt{x}+1\right| + C$$

$$\int \frac{dx}{\sqrt{e^x-1}}$$

$$\sqrt{e^x-1} = u \quad \Rightarrow \quad e^x-1 = u^2 \quad \Rightarrow \quad e^x = u^2+1$$

$$\Rightarrow \quad x = \ln\left(u^2+1\right) \quad \Rightarrow \quad dx = \frac{2u\,du}{u^2+1}$$

$$\int \frac{dx}{\sqrt{e^x-1}} = \int \frac{\frac{2u\,du}{u^2+1}}{u} = 2\int \frac{du}{u^2+1} = 2ang\tan u + C$$

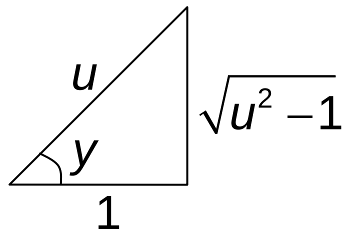
$$\therefore \quad \int \frac{dx}{\sqrt{e^x-1}} = 2ang\tan\sqrt{e^x-1} + C$$

$$\int \frac{\sqrt{x^2+1}}{x^3}dx$$

$$\sqrt{x^2+1}=u \quad \Rightarrow \quad x^2+1=u^2$$

$$\Rightarrow \quad x^2=u^2-1 \quad \Rightarrow \quad x\,dx=u\,du$$

$$\int \frac{\sqrt{x^2+1}}{x^3}dx=\int \frac{\sqrt{x^2+1}}{x^4}x\,dx=\int \frac{u}{\left(u^2-1\right)^2}u\,du=\int \frac{u^2\,du}{\left(u^2-1\right)^2}$$



$$u=\sec y \quad \Rightarrow \quad du=\sec y\tan y\,dy$$

$$\sqrt{u^2-1}=\tan y \quad \Rightarrow \quad \left(u^2-1\right)^2=\tan^4 y$$

$$\int \frac{u^2\,du}{\left(u^2-1\right)^2}=\int \frac{\sec^2 y\sec y\tan y\,dy}{\tan^4 y}$$

$$=\int \frac{\sec^3 y\,dy}{\tan^3 y}=\int \frac{\frac{1}{\cos^3 y}}{\frac{sen^3 y}{\cos^3 y}}dy$$

$$=\int \frac{1}{sen^3}dy=\int \csc^3 y\,dy=\int \csc y\csc^2 y\,dy$$

$$v=\csc y \quad \Rightarrow \quad dv=-\csc y\cot y\,dy$$

$$dw=\csc^2 y\,dy \quad \Rightarrow \quad w=-\cot y$$

$$\int \csc^3 y\,dy=-\csc y\cot y-\int \csc y\cot^2 y\,dy$$

$$\int \csc^3 y\,dy=-\csc y\cot y-\int \csc y\left(\csc^2 y-1\right)dy$$

$$\int \csc^3 y\,dy=-\csc y\cot y-\int \csc^3 y\,dy+\int \csc y\,dy$$

$$2\int \csc^3 y\,dy=-\csc y\cot y+\int \csc y\,dy$$

$$2\int \csc^3 y\,dy=-\csc y\cot y+\ln\left|\csc y+\cot y\right|+C_1$$

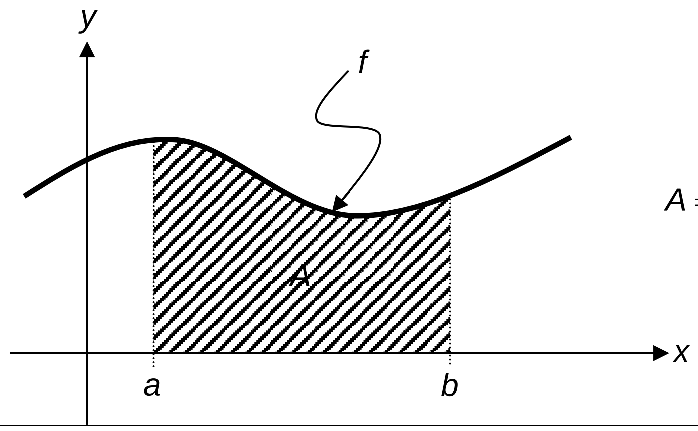
$$\therefore \quad \int \csc^3 y\,dy=-\frac{1}{2}\csc y\cot y+\frac{1}{2}\ln\left|\csc y+\cot y\right|+C$$

$$\int \frac{u^2\,du}{\left(u^2-1\right)^2} = -\frac{1}{2}\frac{u}{\sqrt{u^2-1}}\frac{1}{\sqrt{u^2-1}} + \frac{1}{2}\ln\left|\frac{u}{\sqrt{u^2-1}} + \frac{1}{\sqrt{u^2-1}}\right| + C$$

$$\int \frac{u^2\,du}{\left(u^2-1\right)^2} = -\frac{u}{2\left(u^2-1\right)} + \frac{1}{2}\ln\left|\frac{u+1}{\sqrt{u^2-1}}\right| + C$$

$$\therefore \quad \int \frac{\sqrt{x^2+1}}{x^3}dx = -\frac{\sqrt{x^2+1}}{2x^2} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}+1}{x}\right| + C$$

ÁREA BAJO LA CURVA

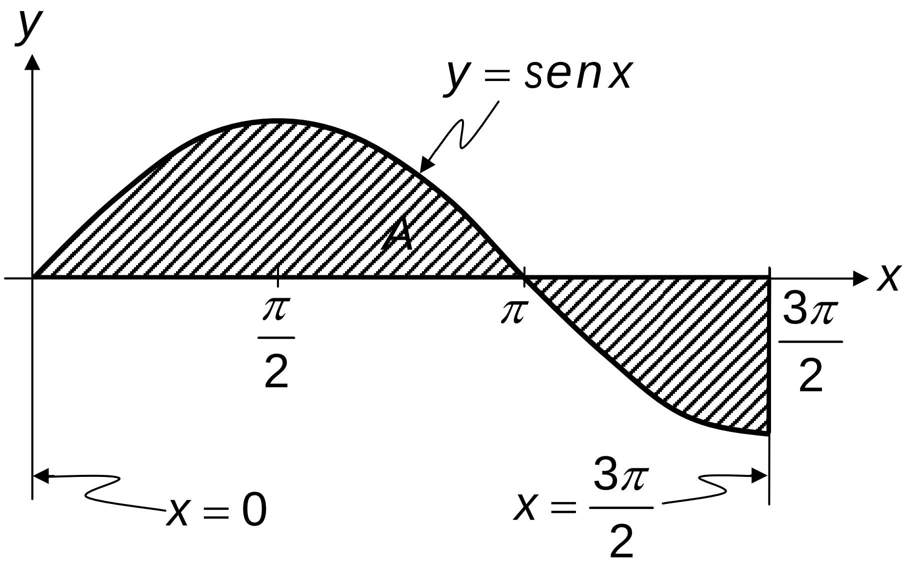


$$A = \int_a^b f(x)\,dx$$

Calcular el valor del área limitada por la gráfica de la función

$$f(x) = sen x,$$

el eje de las abscisas y las rectas $x = 0$ y $x = \frac{3}{2}\pi$



$$A=\int_0^{\pi} sen\,x\,dx-\int_{\pi}^{\frac{3\pi}{2}} sen\,x\,dx$$

$$A=\left[-\cos x\right]_0^{\pi}-\left[-\cos x\right]_{\pi}^{\frac{3\pi}{2}}$$

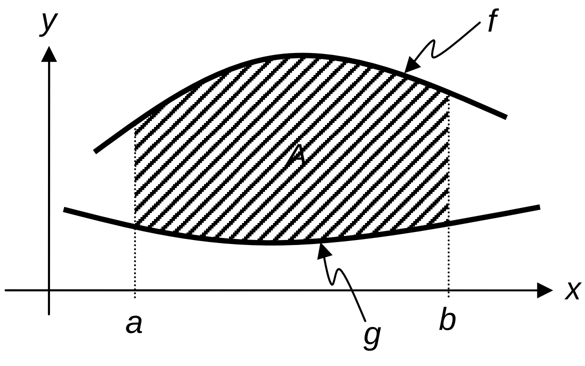
$$=-\cos\pi+\cos 0-\left(-\cos\frac{3\pi}{2}+\cos\pi\right)$$

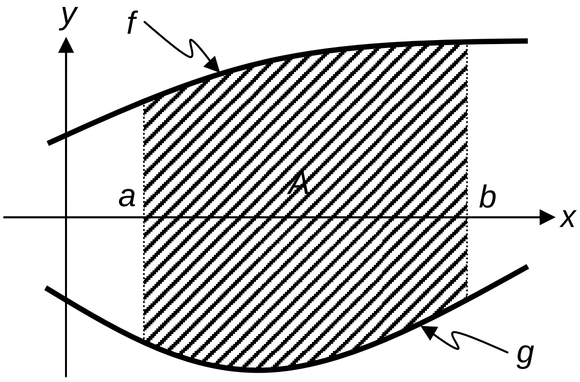
$$A=-\cos\pi+\cos 0+\cos\frac{3\pi}{2}-\cos\pi$$

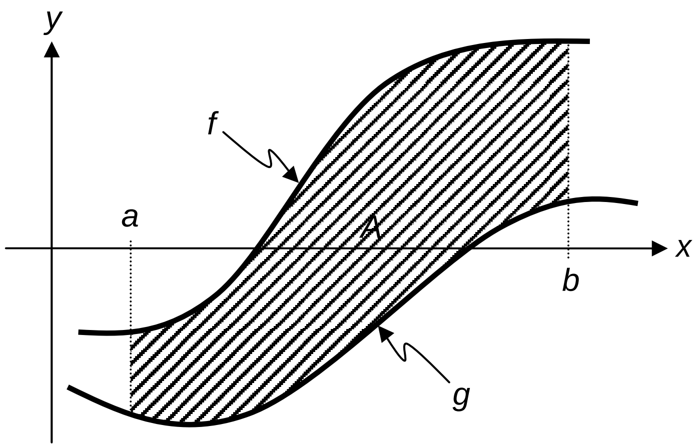
$$=-(-1)+1-0-(-1)=3$$

$$\therefore\quad A=3\,u^2$$

ÁREA ENTRE CURVAS



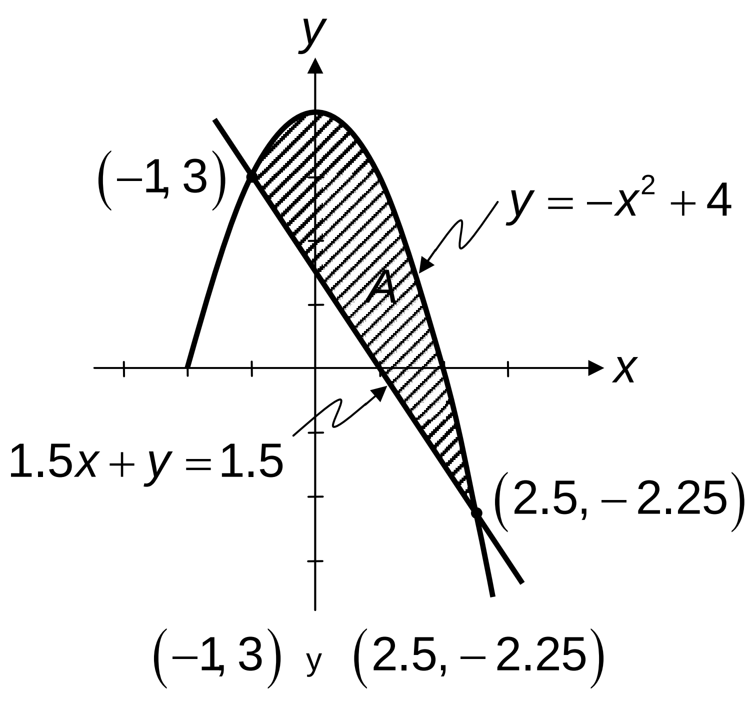




$$A=\int_a^b\left[f(x)-g(x)\right]dx$$

Calcular el valor del área de la región limitada por las curvas:

$$y=-x^2+4 \qquad y \qquad 1.5x+y=1.5$$



$(-1,3)$ y $(2.5,-2.25)$

$$y = f(x) = -x^2 + 4 \qquad y \qquad 1.5x + y = 1.5$$

$$\Rightarrow \quad y = g(x) = 1.5 - 1.5x$$

$$A = \int_{-1}^{2.5} \left[\left(-x^2 + 4 \right) - \left(1.5 - 1.5x \right) \right] dx$$

$$= \int_{-1}^{2.5} \left[\left(-x^2 + 4 \right) - \left(1.5 - 1.5x \right) \right] dx$$

$$A = \int_{-1}^{2.5} \left(-x^2 + 1.5x + 2.5 \right) dx = \left[-\frac{x^3}{3} + \frac{1.5x^2}{2} + 2.5x \right]_{-1}^{2.5}$$

$$A = \left(-\frac{(2.5)^3}{3} + \frac{1.5(2.5)^2}{2} + 2.5(2.5) \right)$$

$$-\left(-\frac{(-1)^3}{3} + \frac{1.5(-1)^2}{2} + 2.5(-1) \right)$$

$$A \approx -5.2083 + 4.6875 + 6.25 - 0.3333 - 0.75 + 2.5$$

$$\therefore \quad A \approx 7.1459\, u^2$$

Calcular el área limitada, en el primer cuadrante, por las gráficas de las curvas:

$$y = x^2 \quad ; \quad y = \frac{x^2}{8} \quad ; \quad y^2 = x \quad ; \quad y^2 = 8x$$

$$\begin{cases} y = x^2 \\ y^2 = x \end{cases} \Rightarrow \quad x^4 = x \quad \Rightarrow \quad x\left(x^3 - 1\right) = 0$$

$$\Rightarrow \quad \begin{cases} x = 0 \quad \Rightarrow \quad y = 0 \\ x = 1 \quad \Rightarrow \quad y = 1 \end{cases}$$

$$\begin{cases} y = x^2 \\ y^2 = 8x \end{cases} \Rightarrow \quad x^4 = 8x \quad \Rightarrow \quad x\left(x^3 - 8\right) = 0$$

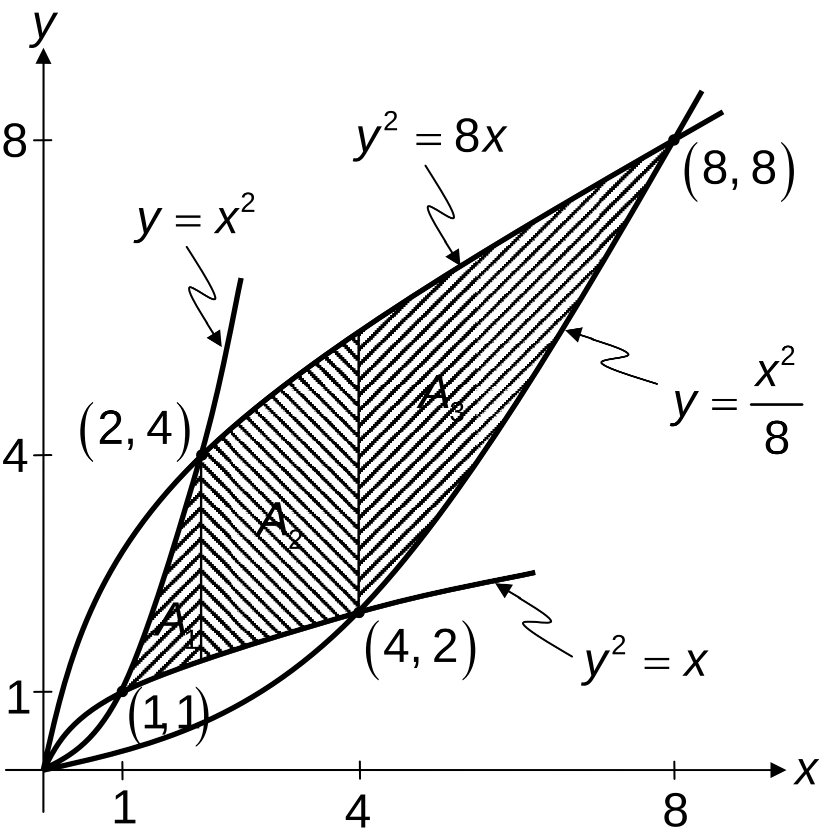
$$\Rightarrow \quad \begin{cases} x = 0 \quad \Rightarrow \quad y = 0 \\ x = 2 \quad \Rightarrow \quad y = 4 \end{cases}$$

$$\begin{cases} y = \dfrac{x^2}{8} \\ y^2 = x \end{cases} \Rightarrow \quad \frac{x^4}{64} = x \quad \Rightarrow \quad x\left(x^3 - 64\right) = 0$$

$$\Rightarrow \quad \begin{cases} x = 0 \quad \Rightarrow \quad y = 0 \\ x = 4 \quad \Rightarrow \quad y = 2 \end{cases}$$

$$\begin{cases} y = \dfrac{x^2}{8} \\ y^2 = 8x \end{cases} \Rightarrow \quad \frac{x^4}{64} = 8x \quad \Rightarrow \quad x\left(x^3 - 512\right) = 0$$

$$\Rightarrow \quad \begin{cases} x = 0 \quad \Rightarrow \quad y = 0 \\ x = 8 \quad \Rightarrow \quad y = 8 \end{cases}$$



$$A_1 = \int_1^2 \left(x^2 - \sqrt{x}\right) dx \; ; \; A_2 = \int_2^4 \left(\sqrt{8x} - \sqrt{x}\right) dx \; ; \; A_3 = \int_4^8 \left(\sqrt{8x} - \frac{x^2}{8}\right) dx$$

$$A_1 = \int_1^2 \left(x^2 - x^{\frac{1}{2}}\right) dx = \left[\frac{x^3}{3} - \frac{2x^{\frac{3}{2}}}{3}\right]_1^2$$

$$=\left(\frac{8}{3}-\frac{4\sqrt{2}}{3}\right)-\left(\frac{1}{3}-\frac{2}{3}\right)=3-\frac{4\sqrt{2}}{3} \qquad \therefore \quad A_1 \approx 1.114\ u^2$$

$$A_2=\int_2^4\left(2\sqrt{2}\ x^{\frac{1}{2}}-x^{\frac{1}{2}}\right)dx=\left[\left(2\sqrt{2}-1\right)\frac{2x^{\frac{3}{2}}}{3}\right]_2^4$$

$$=\left(2\sqrt{2}-1\right)\left(\frac{16}{3}-\frac{4\sqrt{2}}{3}\right) \qquad \therefore \quad A_2 \approx 6.303\ u^2$$

$$A_3=\int_4^8\left(2\sqrt{2}\ x^{\frac{1}{2}}-\frac{x^2}{8}\right)dx=\left[\frac{4\sqrt{2}\ x^{\frac{3}{2}}}{3}-\frac{x^3}{24}\right]_4^8$$

$$=\left(\frac{128}{3}-\frac{64}{3}\right)-\left(\frac{32\sqrt{2}}{3}-\frac{8}{3}\right) \qquad \therefore \quad A_3 \approx 8.915\ u^2$$

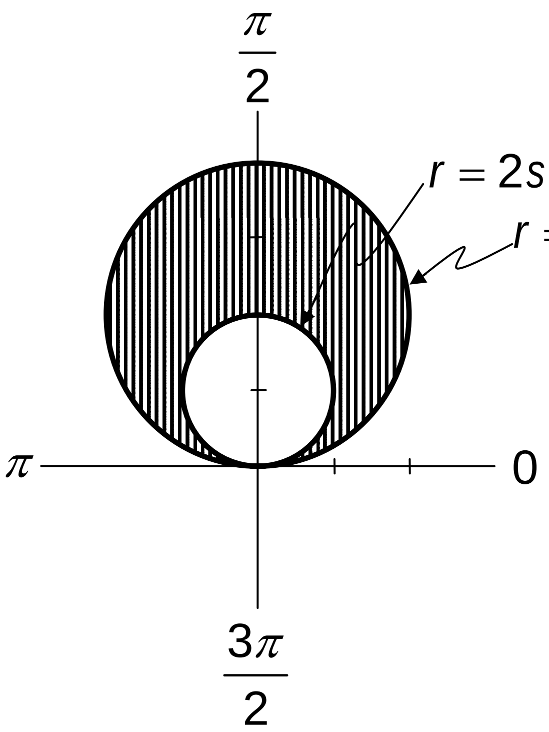
$$A_T=A_1+A_2+A_3 \quad \therefore \quad \therefore \quad A_T=16.332\ u^2$$

ÁREA EN COORDENADAS POLARES

$$A=\frac{1}{2}\int_\alpha^\beta\left[f\left(\theta\right)\right]^2 d\theta=\frac{1}{2}\int_\alpha^\beta r^2\ d\theta$$

Calcular el área limitada por las curvas:

$$r=4sen\theta \quad y \quad r=2sen\theta$$



$$A = \frac{1}{2}\int_0^{\pi} 16sen^2\theta\, d\theta - \frac{1}{2}\int_0^{\pi} 4sen^2\theta\, d\theta = 6\int_0^{\pi} sen^2\theta\, d\theta$$

$$A = 6\int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)d\theta = 6\left[\frac{\theta}{2} - \frac{sen2\theta}{2}\right]_0^{\pi} = 6\left(\frac{\pi}{2}\right) = 3\pi$$

$$\therefore \quad A = 3\pi\, u^2$$

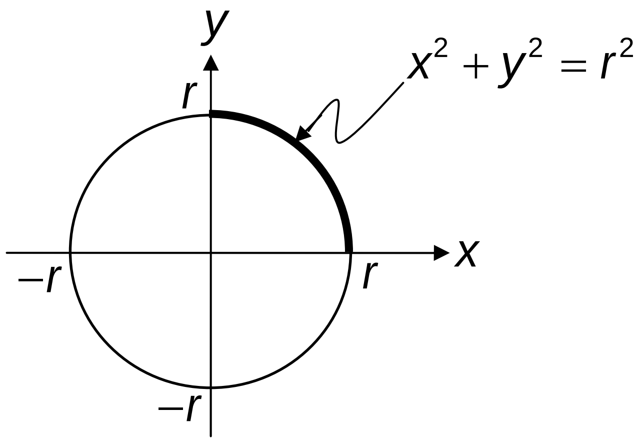
LONGITUD DE ARCO

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta$$

Verificar que la longitud de una circunferencia de radio r es $2\pi r$:
a) Con la expresión que define la longitud de arco cuando la función está expresada en su forma explícita, es decir, $y = f(x)$

b) Mediante la expresión que define la longitud de arco cuando la función está dada por sus ecuaciones paramétricas.



$$y=\sqrt{r^2-x^2} \quad \Rightarrow \quad \frac{dy}{dx}=\frac{-x}{\sqrt{r^2-x^2}}$$

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \Rightarrow \quad L=4\int_0^r\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$$

$$L=4\int_0^1\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx=4\int_0^r\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx$$

$$=4\int_0^r\frac{r}{\sqrt{r^2-x^2}}\,dx=4r\int_0^r\frac{dx}{\sqrt{r^2-x^2}}$$

$$L=4r\left[angsen\,\frac{x}{r}\right]_o^r$$

$$L=4r\left[\left(angsen\,1\right)-\left(angsen\,0\right)\right]=4r\left(\frac{\pi}{2}\right)$$

$$L=2\pi r$$

Ecuaciones paramétricas de la curva:

$$x=r\cos\theta \quad y \quad y=r\,sen\,\theta$$

$$x=r\cos\theta \quad \Rightarrow \quad \frac{dx}{d\theta}=-r\,sen\,\theta$$

$$y=r\,sen\,\theta \quad \Rightarrow \quad \frac{dy}{d\theta}=r\cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

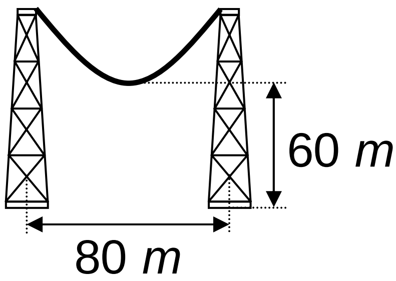
$$\Rightarrow L = 4\int_0^{\frac{\pi}{2}} \sqrt{\left(-r\,\text{sen}\,\theta\right)^2 + \left(r\cos\theta\right)^2}\, d\theta = 4\int_0^{\frac{\pi}{2}} r\, d\theta$$

$$L = 4r\left[\theta\right]_0^{\frac{\pi}{2}} = 2\pi r$$

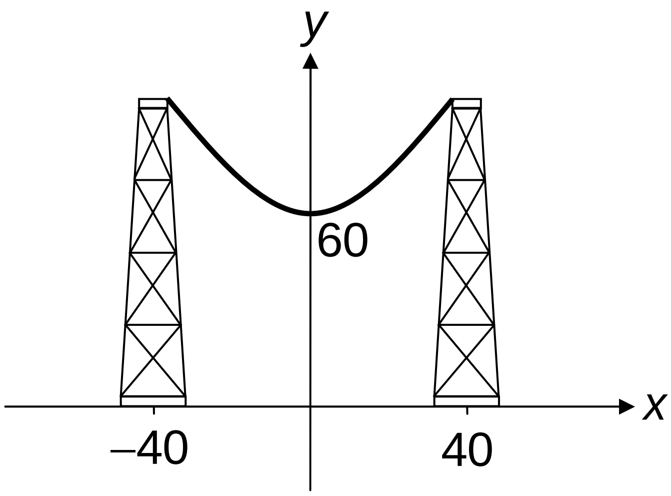
Un cable eléctrico cuelga de dos torres separadas una distancia de 80 m y como se observa en la figura, la forma que adopta el cable es la de la catenaria de ecuación:

$$y = 60\cosh\frac{x}{60}$$

Calcular la longitud de arco de esta catenaria entre las dos torres en las que se apoya.



En un sistema coordenado



$$y = 60\cosh\frac{x}{60} \quad \Rightarrow \quad y = 60\left(\frac{e^{\frac{x}{60}} + e^{-\frac{x}{60}}}{2}\right)$$

$$\Rightarrow \quad y = 30\left(e^{\frac{x}{60}} + e^{-\frac{x}{60}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(e^{\frac{x}{60}} - e^{-\frac{x}{60}}\right) \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{\frac{x}{30}} - 2 + e^{-\frac{x}{30}}\right)$$

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx = 2\int_0^{40} \sqrt{1 + \frac{1}{4}\left(e^{\frac{x}{30}} - 2 + e^{-\frac{x}{30}}\right)}\, dx$$

$$= 2\int_0^{40} \sqrt{\frac{1}{4}\left(e^{\frac{x}{30}} + 2 + e^{-\frac{x}{30}}\right)}\, dx = \int_0^{40} \sqrt{\left(e^{\frac{x}{60}} + e^{-\frac{x}{60}}\right)^2}\, dx$$

$$= \int_0^{40} \left(e^{\frac{x}{60}} + e^{-\frac{x}{60}}\right) dx$$

$$= 60\left[e^{\frac{x}{60}} - e^{-\frac{x}{60}}\right]_0^{40} = 60\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right) = 60\left(e^{\frac{2}{3}} - e^{-\frac{2}{3}}\right)$$

$$= 60\left(e^{\frac{2}{3}} - \frac{1}{e^{\frac{2}{3}}}\right) \approx 60(1.9477 - 0.5134) \approx 86.058\ m$$

Por lo tanto, la longitud del cable es de

$$\therefore \quad L \approx 86.058\ m$$

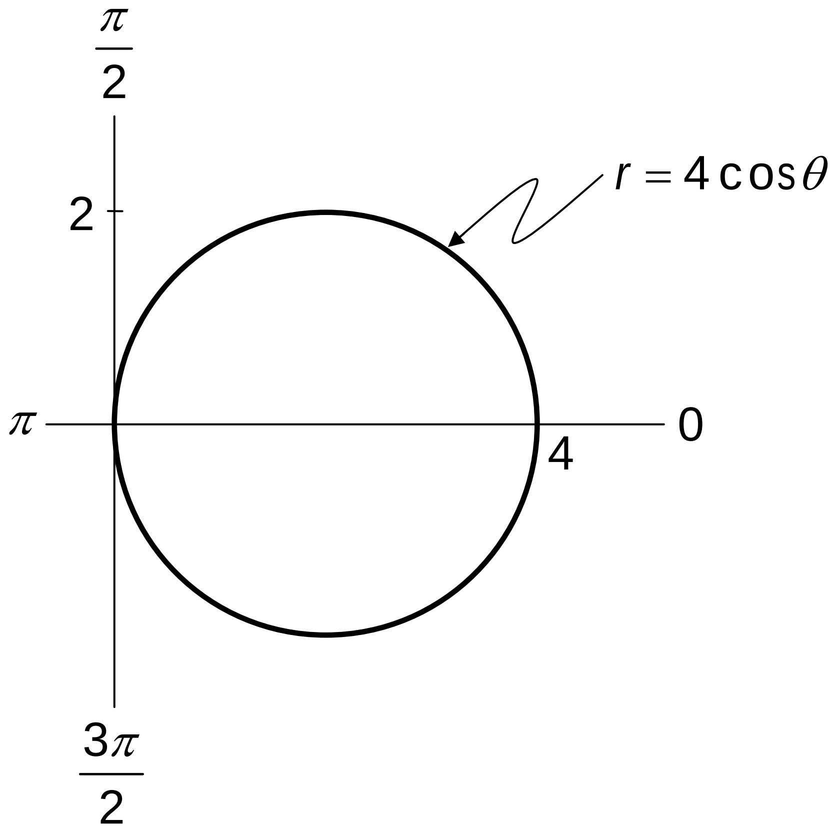
LONGITUD DE ARCO EN COORDENADAS POLARES

$$L = \int_\alpha^\beta \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Calcular la longitud de arco de la gráfica de la función

$$r = 4\cos\theta$$

de $\theta = -\frac{\pi}{2}$ a $\theta = \frac{\pi}{2}$



$$r = 4\cos\theta \quad \Rightarrow \quad r^2 = 4r\cos\theta \quad \Rightarrow \quad x^2 + y^2 = 4x$$

$$\Rightarrow \quad x^2 - 4x + 4 - 4 + y^2 = 0 \quad \Rightarrow \quad (x-2)^2 + y^2 = 4$$

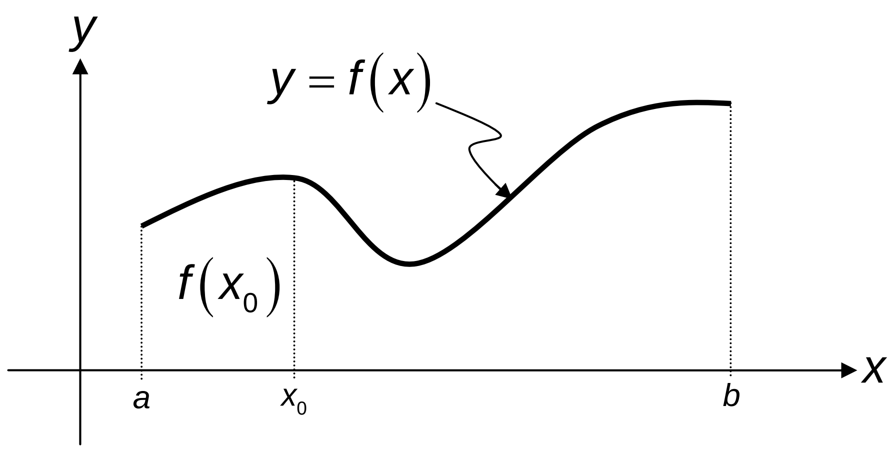
$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\; d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{16\cos^2\theta + 16sen^2\theta}\; d\theta$$

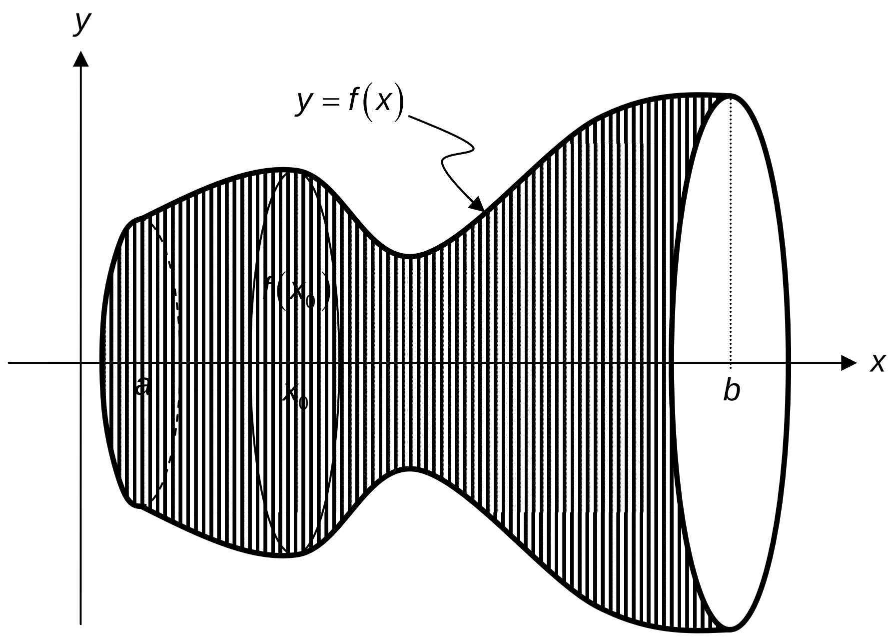
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\; d\theta = 4\left[\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi$$

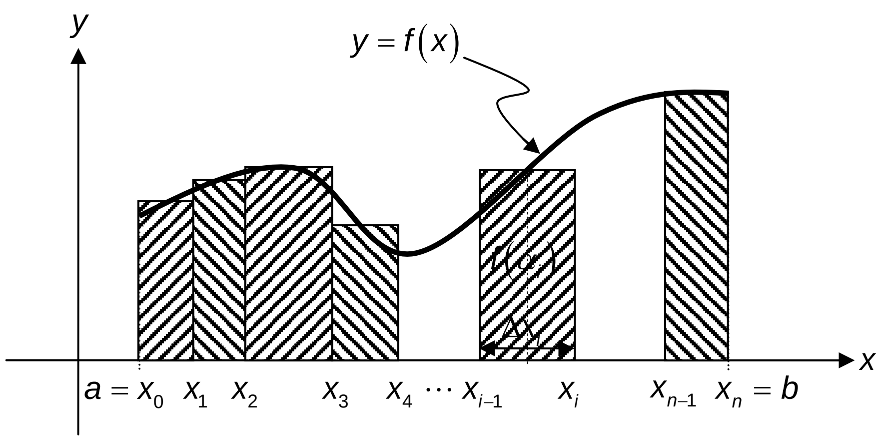
$$\therefore \quad L = 4\pi\ u$$

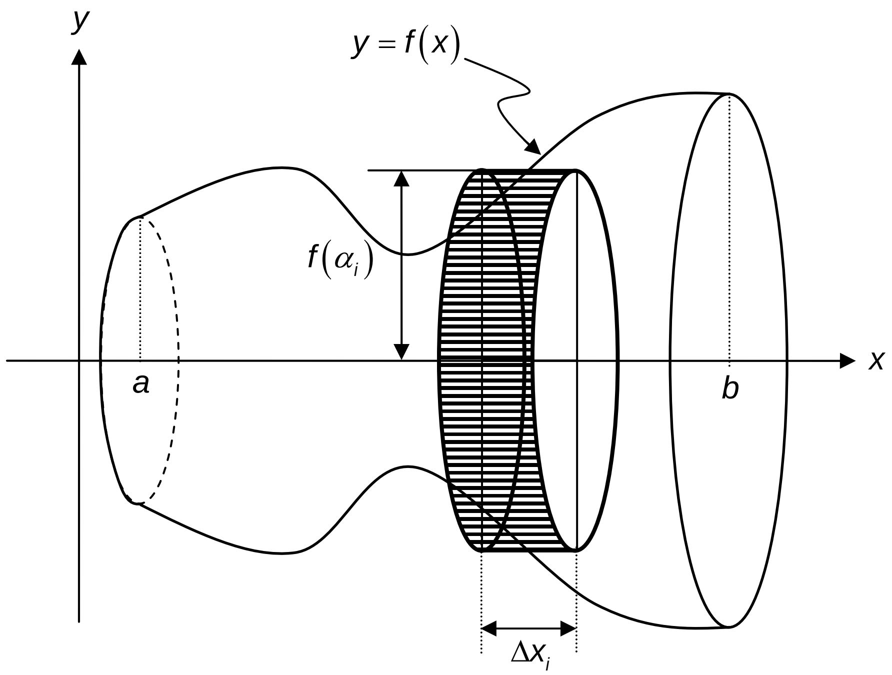
VOLUMEN DE SÓLIDO DE REVOLUCIÓN

MÉTODO DE DSCOS CILÍNDRICOS


y
y = f(x)
f(x0)
x
a
x0
b


y
y = f(x)
x
a
x0
b


y
y = f(x)
x
a = x0 x1 x2 x3 x4 ⋯ xi−1 xi xn−1 xn = b



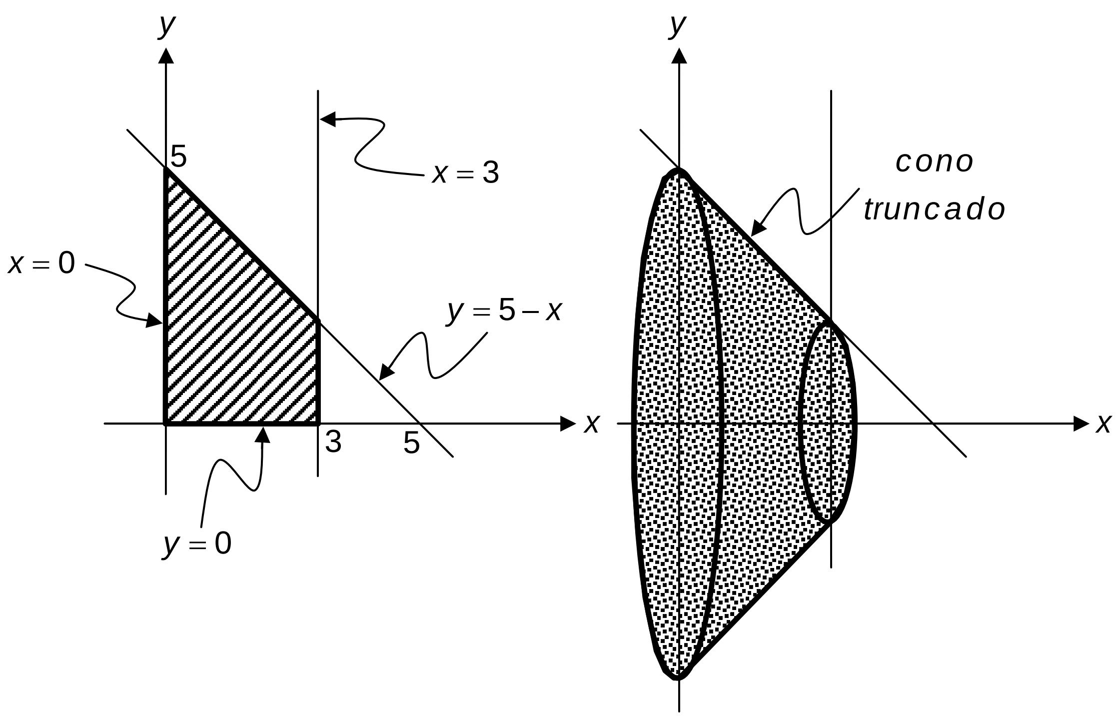
$$V = \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[f(\alpha_i)\right]^2\Delta x_i = \int_a^b \pi\left[f(x)\right]^2 dx = \pi\int_a^b\left[f(x)\right]^2 dx$$

$$V = \int_c^d \pi\left[f(y)\right]^2 dy = \pi\int_c^d\left[f(y)\right]^2 dy$$

Calcular el volumen del cono truncado que se genera al hacer girar, alrededor del eje de las abscisas, a la superficie limitada por las rectas:

$$y = 5 - x \quad ; \quad y = 0 \quad ; \quad x = 0 \quad ; \quad x = 3$$

Hacer un trazo aproximado de la superficie de giro, así como del cono truncado cuyo volumen se pide



El volumen del cono trucado es igual a:

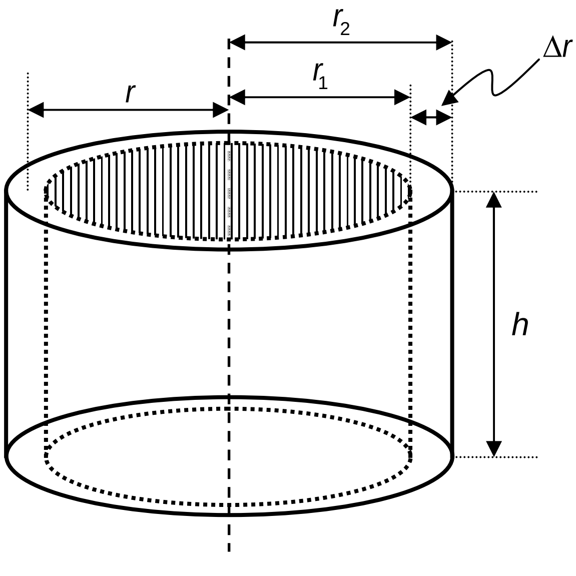
$$V = \pi \int_a^b \left[f(x) \right]^2 dx = \pi \int_0^3 (5-x)^2 dx = \pi \int_0^3 \left(25 - 10x + x^2 \right) dx$$

$$= \pi \left[25x - 5x^2 + \frac{x^3}{2} \right]_0^3 = \pi \left(75 - 45 + \frac{27}{2} \right) = \frac{87}{2} \pi$$

$$\therefore \quad V \approx 136.66 \, u^3$$

MÉTODO DE LAS CORTEZAS CILÍNDRICAS

Este método se basa en utilizar anillos cilíndricos de poco grosor llamados cortezas y que se ilustra en la siguiente figura:



El volumen de una corteza cilíndrica de radio exterior r_2 , radio interior r_1 y altura h está dado por:

$V =$ volumen del cilindro exterior

menos volumen del del hueco

$$V = \pi r_2^2 h - \pi r_1^2 h$$

que también se puede escribir como:

$$V = \pi \left(r_2^2 - r_1^2 \right) h = \pi \left(r_2 + r_1 \right) \left(r_2 - r_1 \right) h = 2\pi \left(\frac{r_2 + r_1}{2} \right) h \left(r_2 - r_1 \right)$$

En el primer paréntesis de la expresión obtenida se tiene el radio medio de la corteza, denotado con r en la figura, es decir, que

$$r = \frac{r_2 + r_1}{2}$$

Y en el segundo paréntesis de dicha expresión se tiene el grosor de la corteza, denotado en la figura con Δr y que equivale a:

$$\Delta r = r_2 - r_1$$

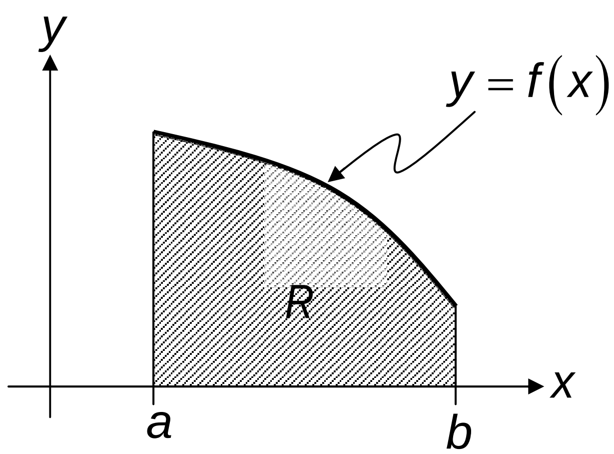
Luego entonces, tomando en consideración esto, el volumen de la corteza cilíndrica se puede escribir como:

$$V = 2\pi\, r\, h \Delta r$$

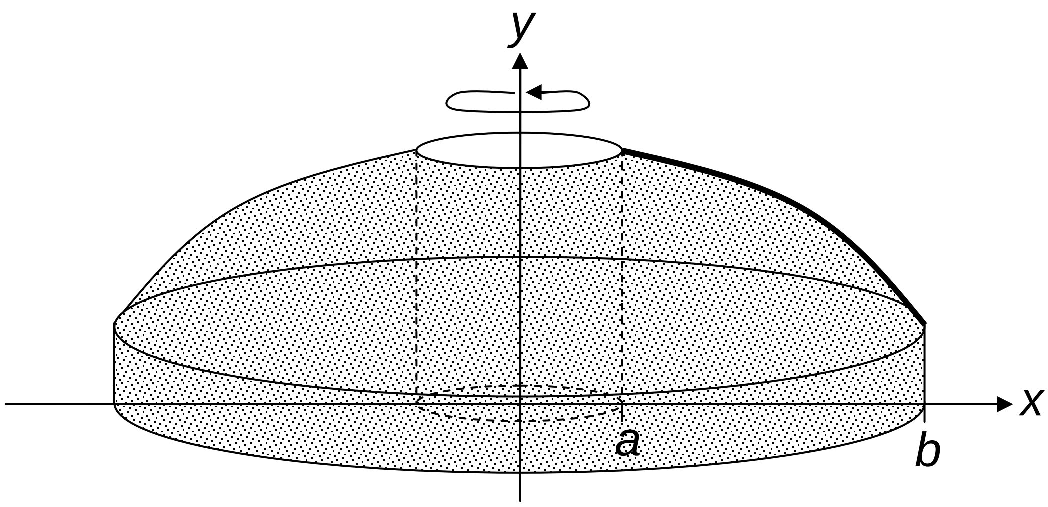
Por lo que.

$$V_{corteza} = 2\pi \left(\text{radio medio}\right)\left(\text{altura}\right)\left(\text{grosor}\right)$$

Sea f una función continua y no negativa en el intervalo cerrado $\left[a,b\right]$, donde $0 \le a < b$. Y sea R la región acotada por la gráfica de la función, el eje de las abscisas y las rectas de ecuaciones $x = a$ y $x = b$, tal como se muestra en la figura siguiente:

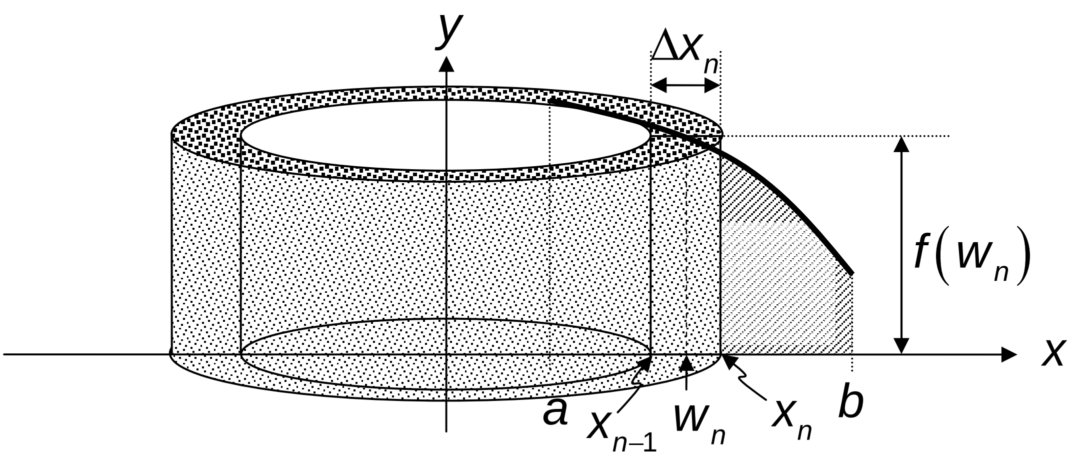


Si se gira la región R alrededor del eje "y" se forma el sólido de revolución mostrado en la figura:



Nótese que si $a > 0$, entonces el sólido de revolución tiene un agujero cilíndrico de radio "a".

Se construye ahora una partición del intervalo $\left[a,b\right]$ y se considera el rectángulo de base $\left[x_{n-1},x_n\right]$ y altura $f\left(w_n\right)$, donde w_n es el punto medio del subintervalo $\left[x_{n-1},x_n\right]$. Cuando este rectángulo gira alrededor del eje "y", entonces se obtiene una corteza cilíndrica con radio medio w_n, altura $f\left(w_n\right)$ y espesor $\Delta x_n = x_n - x_{n-1}$



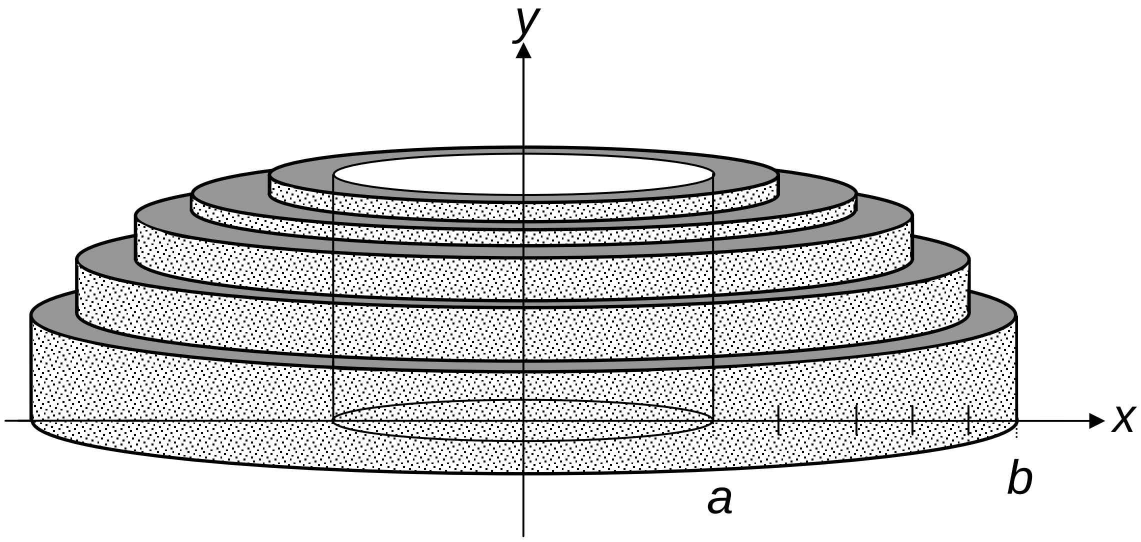
El volumen de esta corteza cilíndrica es.

$$V_n = 2\pi w_n f\left(w_n\right)\Delta x_n$$

Si se suman todos los volúmenes de los subintervalos de la partitcón se llega a:

$$V \approx \sum_n 2 w_n f\left(w_n\right)\Delta x_n$$

que es una suma que proporciona un valor aproximado al volumen del sólido de revolución en cuestión. Una figura aproximada para ilustrar el volumen correspondiente a esta sumatoria se muestra a continuación, considerando solamente una partición con pocos subintervalos.



Es evidente que mientras menor sea la norma $\|\Delta\|$ de la partición, mayor será la aproximación de la sumatoria con el volumen del sólido de revolución, objeto del problema en estudio.

De acuerdo con lo ya estudiado del límite de una sumatoria, es posible establecer la siguiente definición:

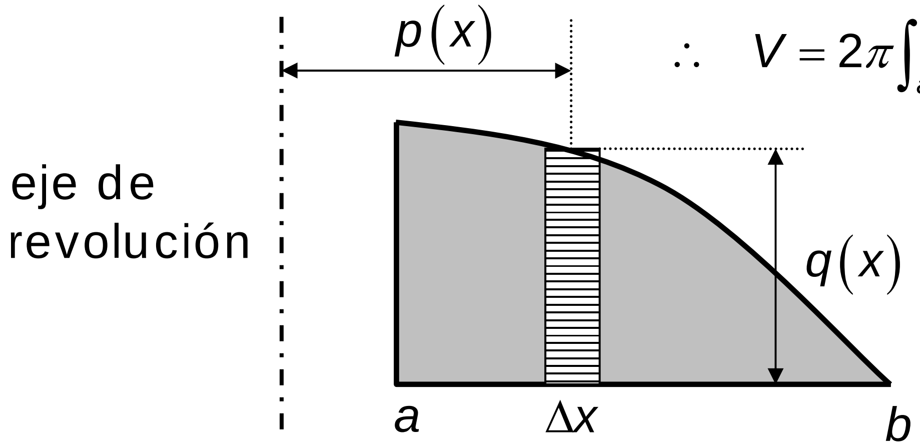
DEFINICIÓN. Sea f una función continua y valuada positivamente en el intervalo $\left[a,b\right]$ para el que se cumple que $0 \le a < b$. Entonces, el volumen V del sólido de revolución que se genera al girar alrededor del eje "y" , la región limitada por la gráfica de f , el eje de las abscisas y las rectas $x=a$ y $x=b$, es igual a:

$$V = \lim_{\|\Delta\|\to 0} \sum_{n} 2\pi\, w_n f\left(w_n\right) \Delta x_k = \int_a^b 2\pi\, x f\left(x\right) dx$$

Como se observa, el volumen se obtiene con una integral definida.

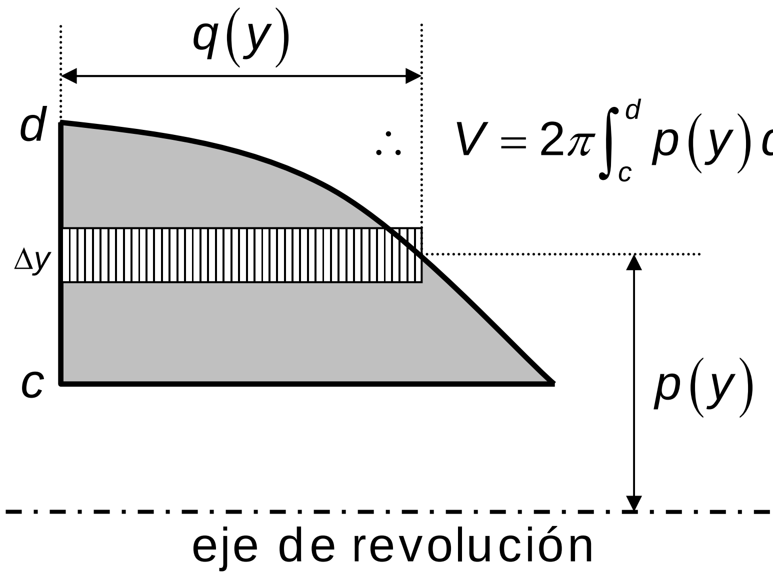
A manera de resumen, considerando las dos posibilidades de ejes de revolución, los ejes "x" y "y", se tiene que:
Se considera una misma región y

i) Si el eje de revolución es el eje vertical , entonces,



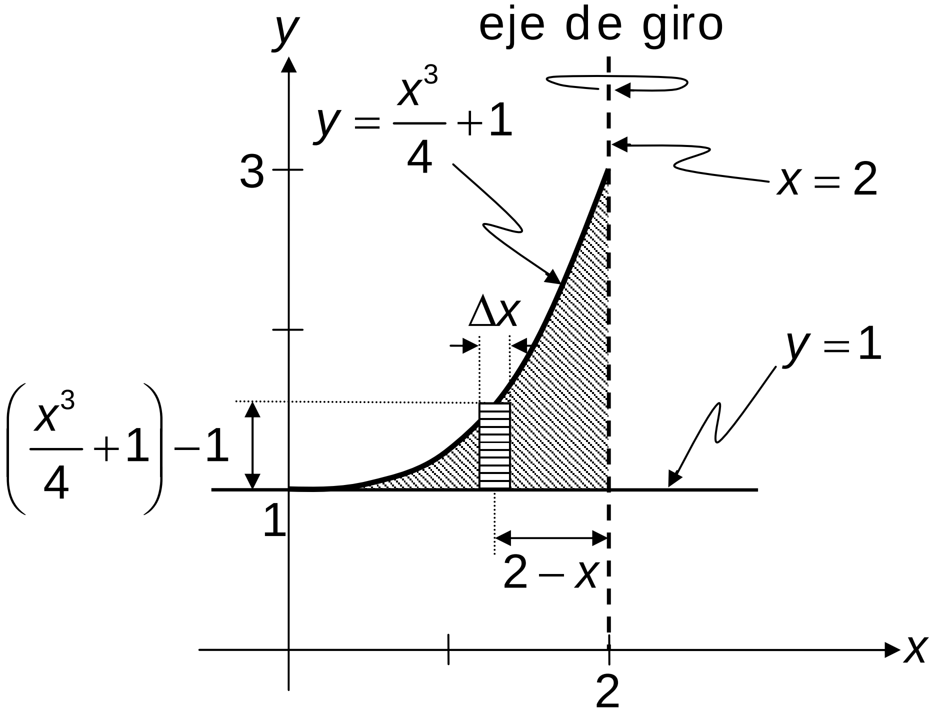
$$\therefore \quad V = 2\pi\int_a^b p(x)q(x)\,dx$$

ii) Si el eje de revolución es el eje horizontal, entonces,



$$\therefore \quad V = 2\pi\int_c^d p(y)q(y)\,dy$$

Calcular el volumen que se genera al girar, alrededor de la recta $x=2$, la región limitada por la curva $y=\dfrac{x^3}{4}+1$ y las rectas $x=2$ y $y=1$. Graficar la región dada y el volumen requerido.



Método de cortezas cilíndricas:

$$\therefore \quad V=2\pi\int_a^b p(x)\,q(x)\,dx$$

$$p(x)=2-x \quad y \quad q(x)=\left(\frac{x^3}{4}+1\right)-1=\frac{x^3}{4}$$

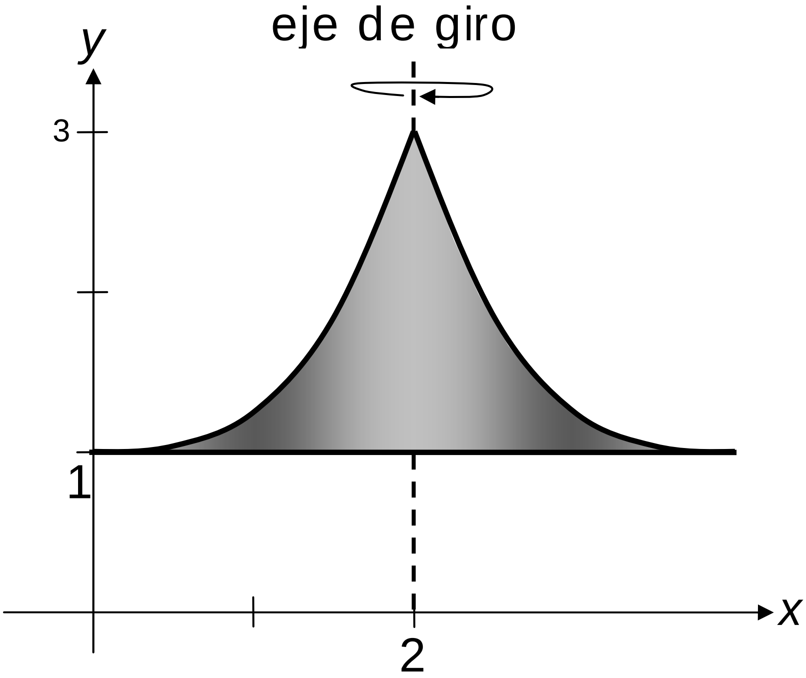
$$V=2\pi\int_0^2 (2-x)\frac{x^3}{4}\,dx$$

$$\int (2-x)\frac{x^3}{4}\,dx=\int\left(\frac{x^3}{2}-\frac{x^4}{4}\right)dx=\frac{x^4}{8}-\frac{x^5}{20}+C$$

$$V=2\pi\left[\frac{x^4}{8}-\frac{x^5}{20}\right]_0^2=2\pi\left(\frac{16}{8}-\frac{32}{20}\right)=2\pi\left(\frac{160-128}{80}\right)$$

$$\therefore \quad V=\frac{4\pi}{5}\,u^3$$

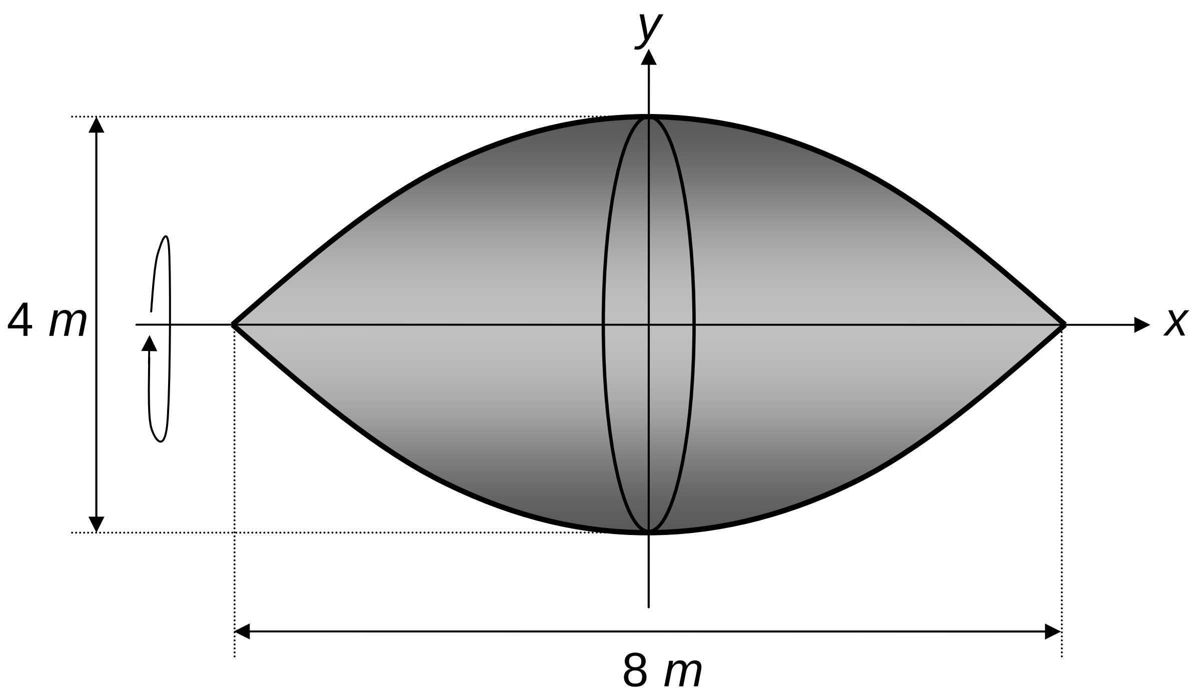
La gráfica de este volumen es:



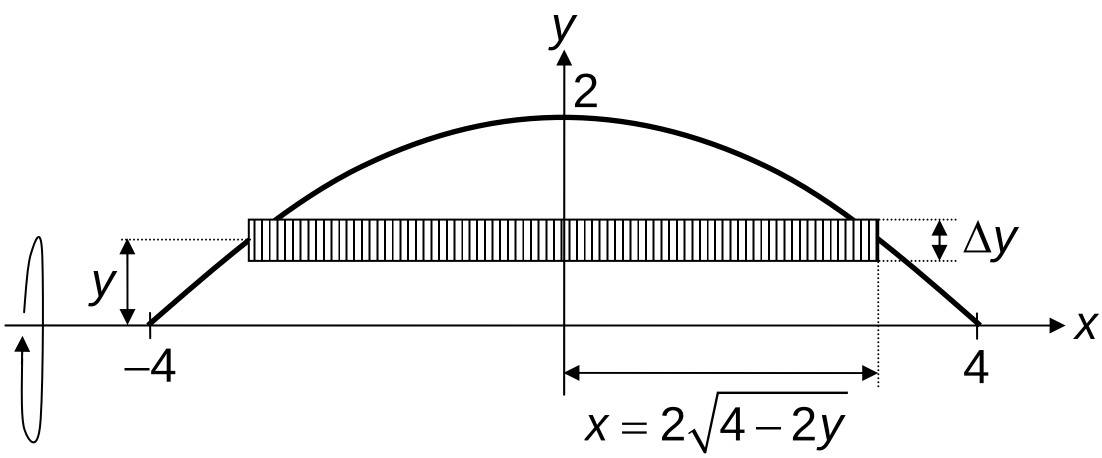
Se construye un depósito de combustible cuya forma se obtiene al hacer girar alrededor del eje de las abscisas, el segmento de la parábola

$$y = 2 - \frac{x^2}{8} \quad ; \quad -4 \le x \le 4$$

¿Cuál es su volumen? (magnitudes "x" y "y" en metros). Utilizar para el cálculo los dos métodos, el de las cortezas cilíndricas y el de los discos



Método de las cortezas



La expresión a utilizar es:

$$V = 2\pi \int_c^d p(y)\, q(y)\, dy$$

$$q(y) = 2x = 4\sqrt{4-2y} \quad ; \quad p(y) = y$$

Luego,

$$V = 2\pi \int_0^2 y\left(4\sqrt{4-2y}\right) dy \quad \Rightarrow \quad V = 8\pi \int_0^2 y\sqrt{4-2y}\ dy$$

Se la integral indefinida y,

$$\int y\sqrt{4-2y}\ dy \quad ; \quad u = 4-2y$$

$$\Rightarrow \quad du = -2\ dy \quad ; \quad y = \frac{4-u}{2}$$

$$-\frac{1}{2}\int \frac{4-u}{2}\sqrt{u}\,du =$$

$$-\frac{1}{4}\int\left(4\sqrt{u}-u\sqrt{u}\right)du = -\frac{1}{4}\int\left(4u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du$$

$$= -\frac{1}{4}\left(\frac{4u^{\frac{3}{2}}}{\frac{3}{2}}-\frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right)+C = -\frac{2}{3}u^{\frac{3}{2}}+\frac{1}{10}u^{\frac{5}{2}}+C =$$

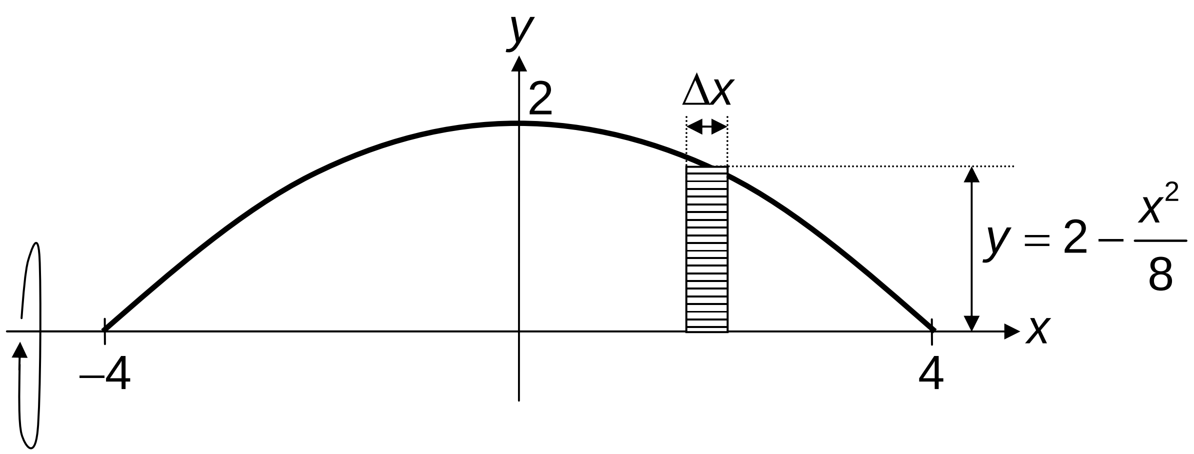
$$-\frac{2}{3}(4-2y)^{\frac{3}{2}}+\frac{1}{10}(4-2y)^{\frac{5}{2}}+C$$

$$V = 8\pi\left[-\frac{2}{3}(4-2y)^{\frac{3}{2}}+\frac{1}{10}(4-2y)^{\frac{5}{2}}\right]_0^2$$

$$= 8\pi\left[\frac{2}{3}(4)^{\frac{3}{2}}-\frac{1}{10}(4)^{\frac{5}{2}}\right]$$

$$V = 8\pi\left(\frac{16}{3}-\frac{32}{10}\right) \quad \therefore \quad V \approx 53.62\ m^3$$

Método de los discos



$$V = \pi\int_{-4}^{4}\left(2-\frac{x^2}{8}\right)^2 dx = 2\pi\int_{0}^{4}\left(2-\frac{x^2}{8}\right)^2 dx$$

$$\int\left(2-\frac{x^2}{8}\right)^2 dx = \int\left(4-\frac{x^2}{2}+\frac{x^4}{64}\right)dx = 4x-\frac{x^3}{6}+\frac{x^5}{320}+C$$

$$V = 2\pi\left[4x - \frac{x^3}{6} + \frac{x^5}{320}\right]_0^4$$

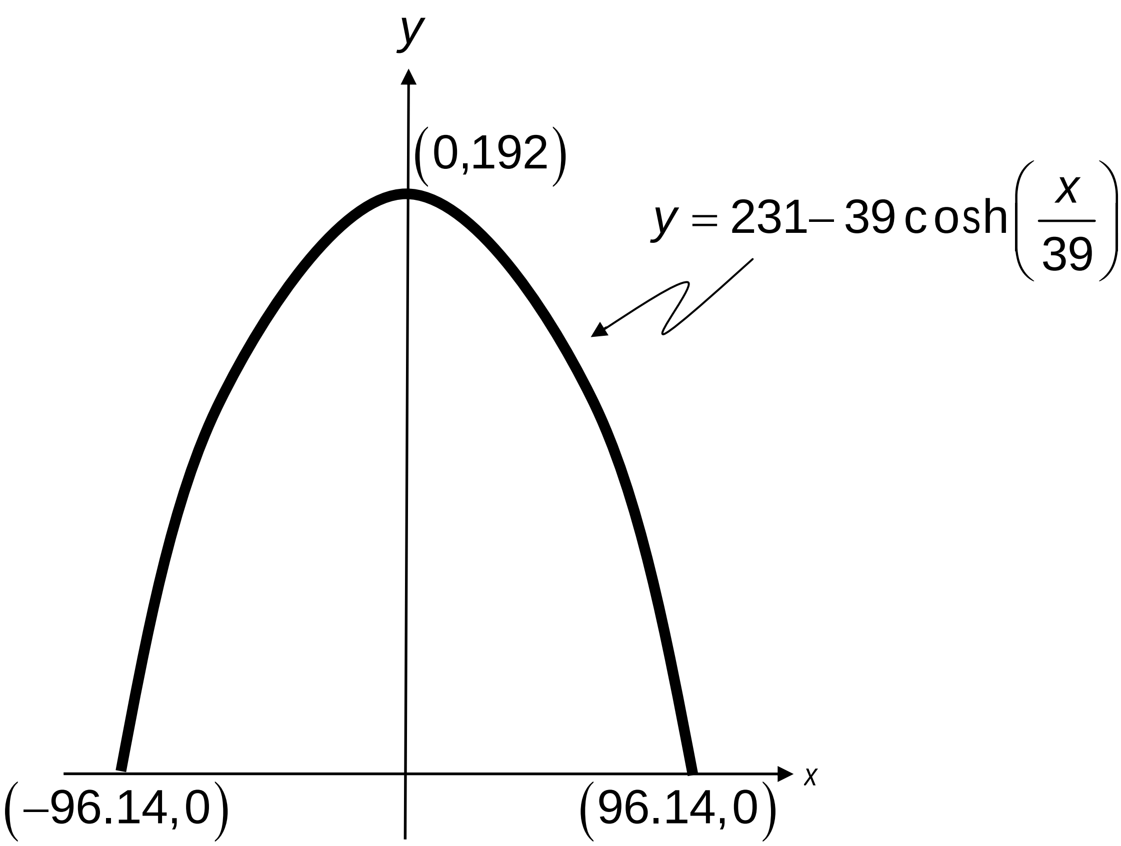
$$= 2\pi\left(16 - \frac{64}{6} + \frac{1024}{320}\right) \quad \therefore \quad V \approx 53.62\ m^3$$

LONGITUD DE ARCO Y ÁREA BAJO LA CURVA

En la ciudad de San Luis Missouri, EUA, se construyó un arco que posee la forma de una catenaria invertida. En el centro tiene 192 m de altura y de extremo a extremo en la base hay una longitud de 192.28 m. La forma del arco obedece, en forma aproximada, a la curva de ecuación:

$$y = 231 - 39\cosh\left(\frac{x}{39}\right)$$

Determinar la longitud total del arco, así como el área total bajo el arco.



Longitud del arco:

$$L = 2\int_0^{96.14} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$y = 231 - 39\cosh\left(\frac{x}{39}\right) \quad ; \quad \frac{dy}{dx} = -senh\left(\frac{x}{39}\right)$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 = senh^2\left(\frac{x}{39}\right)$$

$$L = 2\int_0^{96.14} \sqrt{1+senh^2\left(\frac{x}{39}\right)} =$$

$$\int_0^{96.14} \cosh\left(\frac{x}{39}\right) dx = 2\left[39senh\left(\frac{x}{39}\right)\right]_0^{96.14} \quad \therefore \quad L \approx 455.52\ m$$

Área bajo la curva:

$$A = 2\int_0^{96.14} f(x)\,dx$$

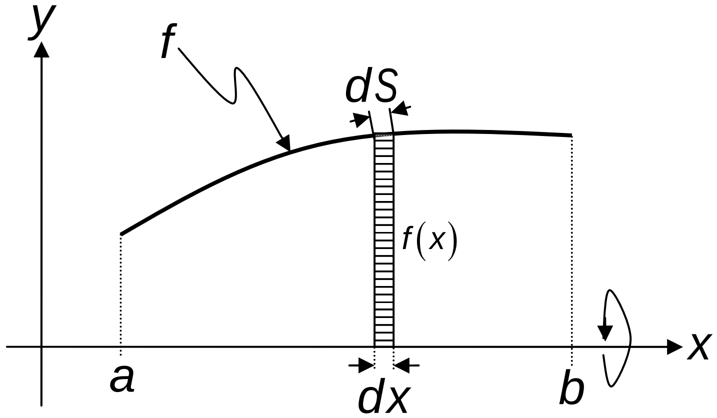
de donde

$$A = 2\int_0^{96.14}\left[231 - 39\cosh\left(\frac{x}{39}\right)\right]dx =$$

$$= 2\left[231x - 39^2 senh\left(\frac{x}{39}\right)\right]_0^{96.14}$$

$$= 2\left(22,208.34 - 8,882.63\right) \quad \therefore \quad A = 26,651.42\ m^2$$

ÁREA DE UNA SUPERFICIE DE REVOLUCIÓN



$$dA = 2\pi f(x)\, dS$$

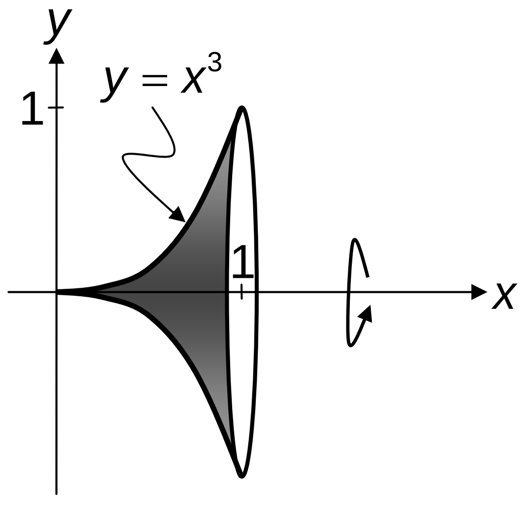
$$A = 2\pi \int_a^b f(x)\, dS$$

$$dS = \sqrt{1+\left[f'(x)\right]^2}\, dx$$

$$A = 2\pi \int_a^b f(x)\, \sqrt{1+\left[f'(x)\right]^2}\, dx$$

Curva	Eje de revolución	Eje de revolución
	"x"	"y"
$y=f(x)$ $a \le x \le b$	$A(S)=2\pi\int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\, dx$	$A(S)=2\pi\int_a^b x\sqrt{1+\left[f'(x)\right]^2}\, dx$
$x=f(y)$ $c \le y \le d$	$A(S)=2\pi\int_c^d y\sqrt{1+\left[g'(y)\right]^2}\, dy$	$A(S)=2\pi\int_c^d g(y)\sqrt{1+\left[g'(y)\right]^2}\, dy$

EJEMPLO. Calcular de dos maneras el área de la superficie que se genera al hacer girar la gráfica de la función $y=f(x)=x^3$, en el intervalo $\left[0,1\right]$, alrededor del eje de las abscisas. Hacer un trazo aproximado de la gráfica de la curva y de la superficie que se genera.



Primera forma:

$$A(S) = 2\pi \int_a^b f(x)\sqrt{1+\left[f'(x)\right]^2}\, dx$$

$$f(x)=x^3 \quad \Rightarrow \quad f'(x)=3x^2$$

$$\Rightarrow \quad A(S)=2\pi\int_0^1 x^3\sqrt{1+9x^4}\ dx$$

$$u=1+9x^4 \quad \Rightarrow \quad du=36x^3dx$$

$$\frac{\pi}{18}\int\sqrt{u}du=\frac{\pi}{18}\frac{2u^{\frac{3}{2}}}{3}+C=\frac{\pi}{27}\left(1+9x^4\right)^{\frac{3}{2}}+C$$

$$A(S)=\frac{\pi}{27}\left[\left(1+9x^4\right)^{\frac{3}{2}}\right]_0^1 \quad \Rightarrow \quad A(S)=\frac{\pi}{27}\left(10^{\frac{3}{2}}-1\right)$$

$$\therefore \quad A(S)\doteq 3.563\,u^2$$

Segunda forma:

$$A(S)=2\pi\int_0^1 y\sqrt{1+\left[g'(y)\right]^2}\ dy$$

$$g(y)=y^{\frac{1}{3}} \quad ; \quad g'(y)=\frac{1}{3}y^{-\frac{2}{3}} \quad \Rightarrow \quad g'(y)=\frac{1}{3y^{\frac{2}{3}}}$$

$$A(S)=2\pi\int_0^1 y\sqrt{1+\frac{1}{9y^{\frac{4}{3}}}}\ dy=2\pi\int_0^1 y\sqrt{\frac{9y^{\frac{4}{3}}+1}{9y^{\frac{4}{3}}}}\ dy$$

$$=2\pi\int_0^1\frac{y}{3y^{\frac{2}{3}}}\sqrt{9y^{\frac{4}{3}}+1}\,dy=\frac{2\pi}{3}\int_0^1 y^{\frac{1}{3}}\sqrt{9y^{\frac{4}{3}}+1}\,dy$$

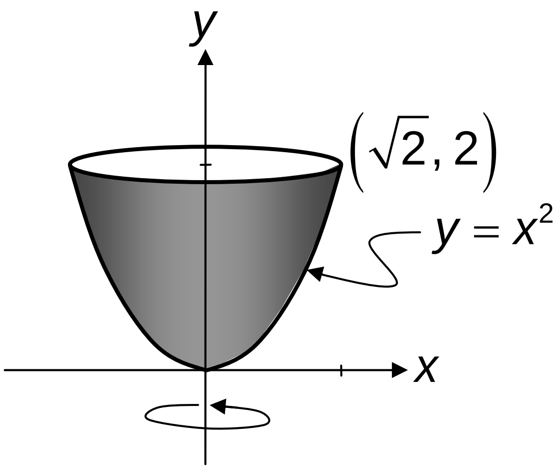
$$u=9y^{\frac{4}{3}}+1 \quad \Rightarrow \quad du=12y^{\frac{1}{3}}dy$$

$$\frac{2\pi}{3\cdot 12}\int\sqrt{u}\ du=\frac{\pi}{18}\frac{2u^{\frac{3}{2}}}{3}+C=\frac{\pi}{27}\left(9y^{\frac{4}{3}}+1\right)^{\frac{3}{2}}+C$$

$$A(S)=\frac{\pi}{27}\left[\left(9y^{\frac{4}{3}}+1\right)^{\frac{3}{2}}\right]_0^1 \quad \Rightarrow \quad A(S)=\frac{\pi}{27}\left(10^{\frac{3}{2}}-1\right)$$

$$\therefore \quad A(s) \doteq 3.563\, u^2$$

Calcular, de dos formas, el área de la superficie generada al girar la curva, gráfica de la función $y = f(x) = x^2$, en el intervalo $x \in \left[0, \sqrt{2}\right]$, alrededor del eje de las ordenadas. Hacer un trazo aproximado de la curva, así como de la superficie que se genera.



Primera forma: Es mediante la expresión

$$A(S) = \int_0^{\sqrt{2}} 2\pi x \sqrt{1 + \left[f'(x)\right]^2}\; dx$$

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x \quad ; \quad A(S) = 2\pi \int_0^{\sqrt{2}} x\sqrt{1 + 4x^2}\; dx$$

$$u = 1 + 4x^2 \quad \Rightarrow \quad du = 8x\, dx$$

$$\frac{\pi}{4}\int \sqrt{u}\; du = \frac{\pi}{4}\frac{2u^{\frac{3}{2}}}{3} + C = \frac{\pi}{6}\left(1 + 4x^2\right)^{\frac{3}{2}} + C$$

$$A(S) = \frac{\pi}{6}\left[\left(1 + 4x^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{2}} \quad \Rightarrow \quad A(S) = \frac{\pi}{6}(27 - 1) \quad \Rightarrow \quad A(S) = \frac{13\pi}{3}$$

$$\therefore \quad A(S) \doteq 13.614\, u^2$$

Segunda forma:

$$A(S) = 2\pi \int_0^2 g(y)\sqrt{1 + \left[g'(y)\right]^2}\; dy$$

$$g(y)=\sqrt{y} \quad \Rightarrow \quad g'(y)=\frac{1}{2\sqrt{y}}$$

$$A(S)=2\pi\int_0^2 \sqrt{y}\sqrt{1+\frac{1}{4y}}\ dy$$

$$A(S)=2\pi\int y^{\frac{1}{2}}\sqrt{\frac{4y+1}{4y}}\ dy=\pi\int\sqrt{4y+1}\,dy$$

$$u=4y+1 \quad \Rightarrow \quad du=4\ dy$$

$$\frac{\pi}{4}\int\sqrt{u}\ du=\frac{\pi}{4}\frac{2u^{\frac{3}{2}}}{3}+C=\frac{\pi}{6}(4y+1)^{\frac{3}{2}}+C$$

$$A(S)=\frac{\pi}{6}\left[(4y+1)^{\frac{3}{2}}\right]_0^2 \quad \Rightarrow \quad A(S)=\frac{\pi}{6}(27-1)$$

$$\therefore \quad A(S)\doteq 13.614\,u^2$$