“Sonido, ecuaciones en derivadas parciales y series de Fourier.

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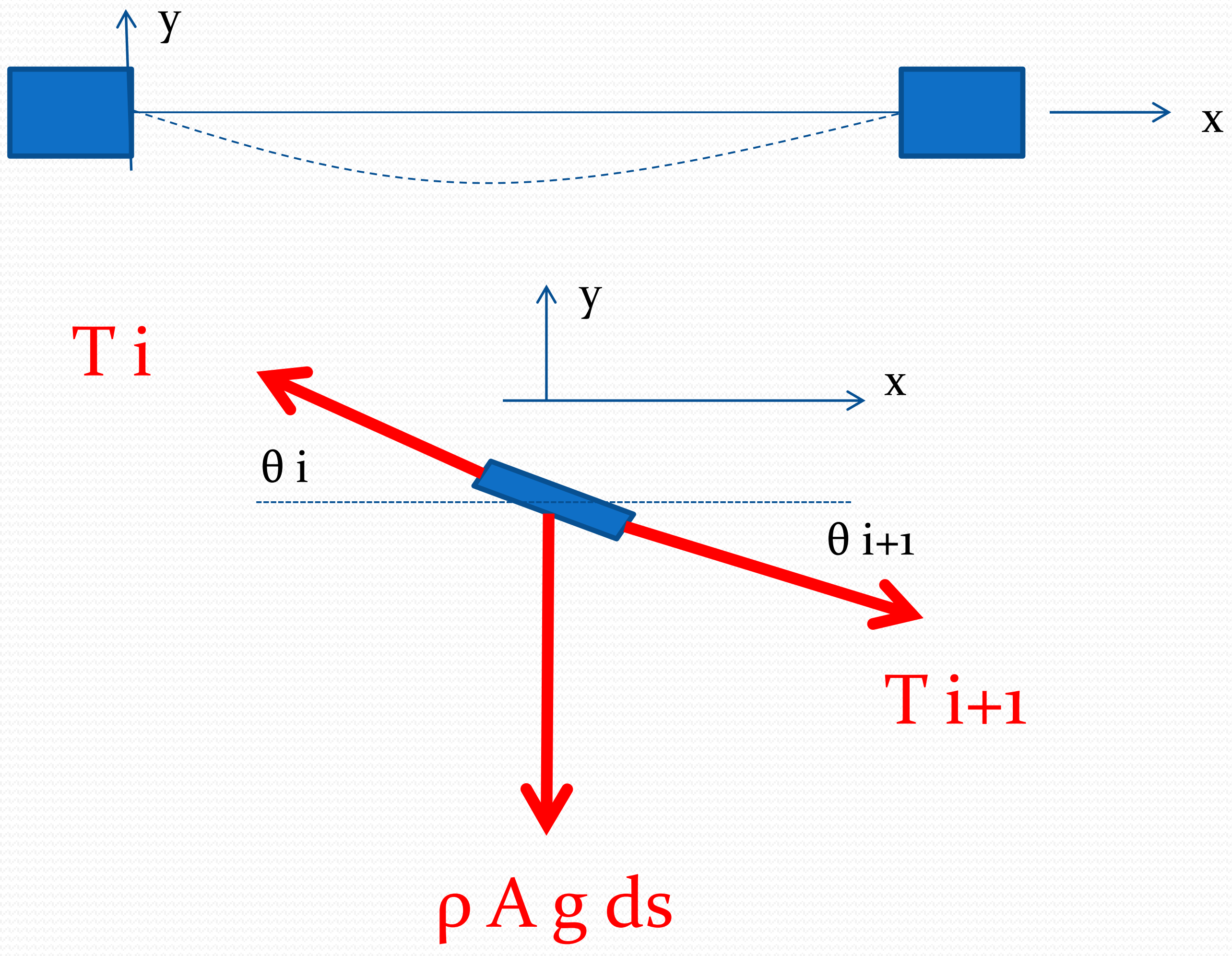
https://www.youtube.com/watch?v=r2Xdlgii-Rc

Contenido

- Ecuación de onda.
- El diapasón y la viga en voladizo.
- Modo de vibración.
- Frecuencias de las notas musicales.
- Términos de la serie de Fourier para diferentes instrumentos musicales.

La ecuación de onda.

Una cuerda con carga distribuida



$$-T_i Cos(\theta_i) + T_{i+1}\, Cos(\theta_{i+1}) = 0$$

$$-T_{i+1} Sin(\theta_{i+1}) + T_i Sin(\theta_i) - g\, \rho\, A\, ds = 0$$

$$T_{i+1} Cos(\theta_{i+1}) = T_i Cos(\theta_i) = T_h$$

$$-\frac{g\, \rho\, A}{Th}\, ds = Tan(\theta_{i+1}) - Tan(\theta_i)$$

$$-\frac{g\,\rho\,A}{Th}\,ds = \frac{dy}{dx}_{i+1} - \frac{dy}{dx}_{i}$$

$$-\frac{g\,\rho\,A}{Th}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \frac{dy}{dx}_{i+1} - \frac{dy}{dx}_{i}$$

$$-\frac{g\,\rho\,A}{Th}\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{d^2y}{dx^2}$$

$$-\frac{g\,\rho\,A}{Th}\sqrt{1+\left(\frac{dy}{dx}\right)^{2}}=\frac{d^{2}y}{dx^{2}}$$

$$-\frac{g\,\rho\,A}{Th}\sqrt{1+u^{2}}=\frac{du}{dx}$$

$$-\frac{g\,\rho\,A}{Th}dx=\frac{du}{\sqrt{1+u^{2}}}$$

$$-\frac{g\,\rho\,A}{Th}x+C1=ArcSinh(u)$$

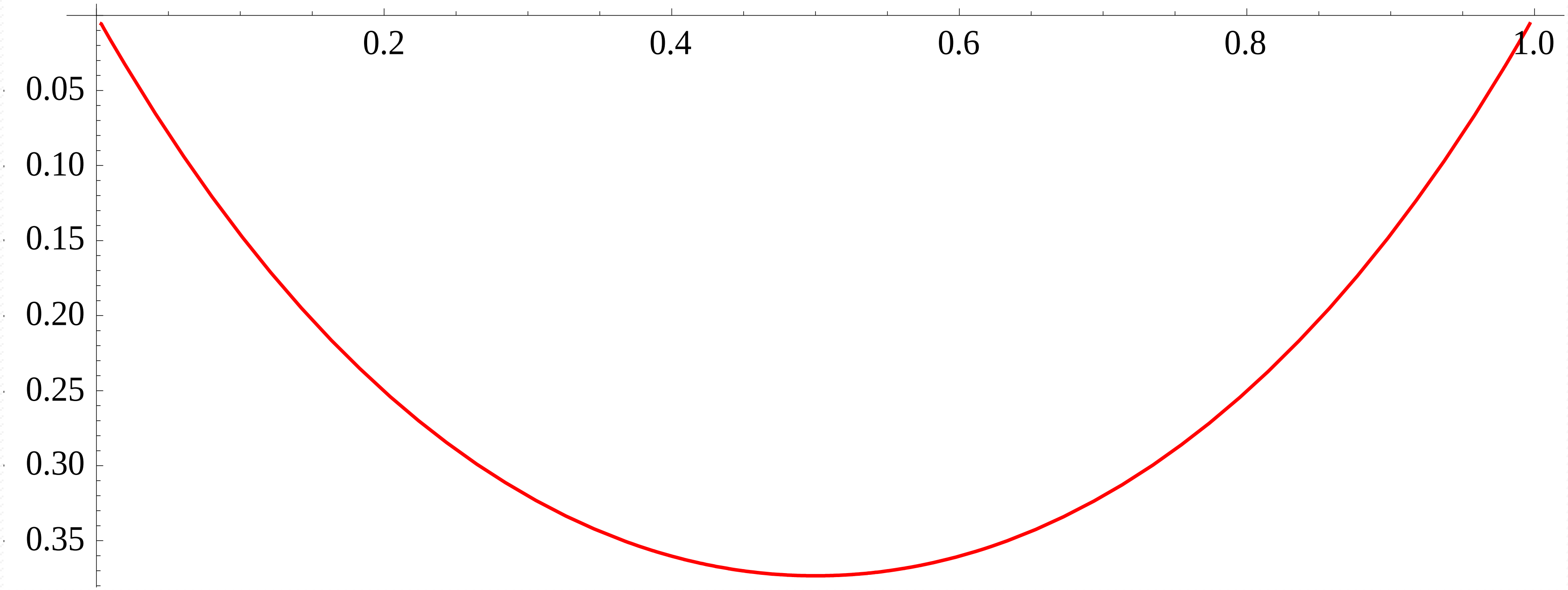
$$u = Sinh\left(-\frac{g\ \rho\ A}{Th}x + C1\right)$$

$$\frac{dy}{dx} = Sinh\left(-\frac{g\ \rho\ A}{Th}x + C1\right)$$

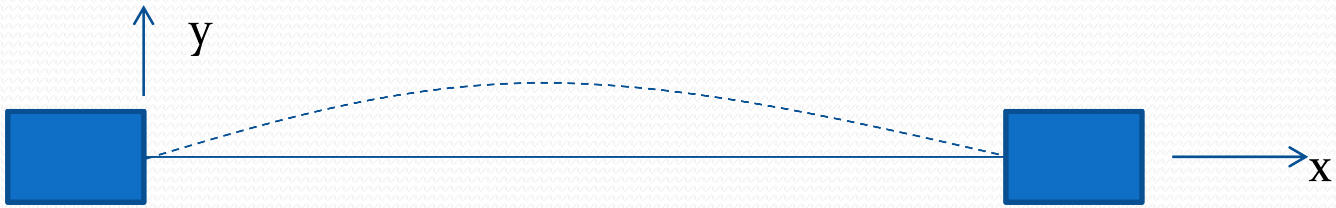
$$y = C2 + \frac{\mathrm{Th}}{Ag\rho}\ \mathrm{Cosh}\left(\frac{A\ g\ \rho}{\mathrm{Th}}x - C1\right)$$

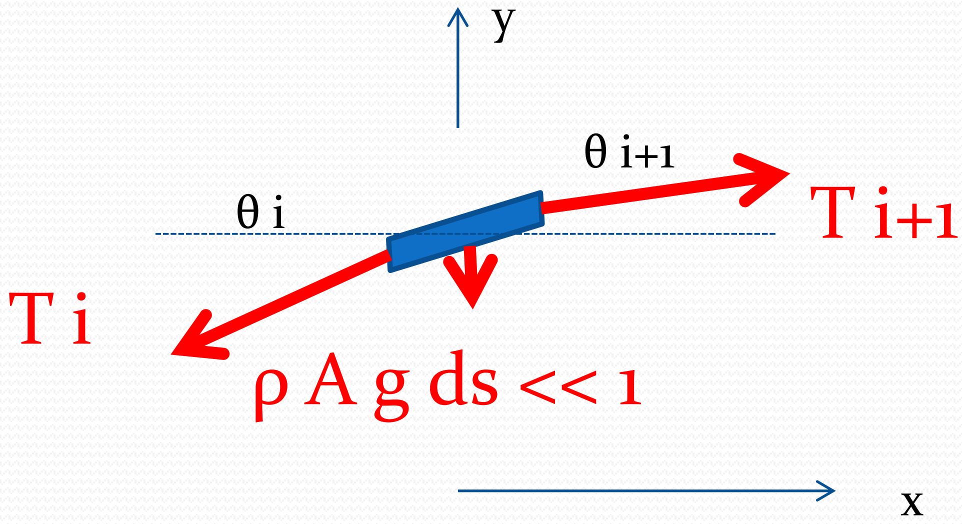
Con condiciones de frontera se puede encontrar las constantes de la solución particular.

$$y[0] = 0, \qquad y[L] = 0$$



Una cuerda fuera del equilibrio





$$-T_i Cos(\theta_i) + T_{i+1}\, Cos(\theta_{i+1}) = 0$$

$$T_{i+1} Cos(\theta_{i+1}) = T_i Cos(\theta_i) = T_h$$

$$T_{i+1} Sin(\theta_{i+1}) - T_i Sin(\theta_i) - g\,\rho\,A\,dx = \rho\,A\,dx \frac{d^2y}{dt^2}$$

$$Tan(\theta_{i+1}) - Tan(\theta_i) - \frac{g\,\rho\,A}{Th} dx = \frac{\rho\,A}{Th}\, dx \frac{d^2y}{dt^2}$$

$$Tan(\theta_{i+1}) - Tan(\theta_i) = \frac{\rho\, A}{Th}\, dx \frac{d^2 y}{dt^2} + \frac{g\,\rho\, A}{Th} dx$$

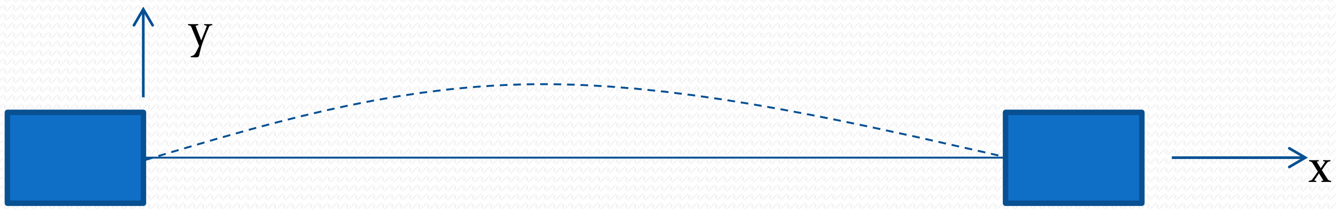
$$\frac{dy}{dx}_{i+1} - \frac{dy}{dx}_i = \left(\frac{\rho\, A}{Th} \frac{d^2 y}{dt^2} - \frac{g\,\rho\, A}{Th} \right) dx$$

$$\frac{d^2 y}{dx^2} = \frac{\rho\, A}{Th} \frac{d^2 y}{dt^2} - \frac{g\,\rho\, A}{Th}$$

$$\frac{d^2y}{dx^2} = \frac{\rho\ A}{Th}\frac{d^2y}{dt^2} - \frac{g\ \rho\ A}{Th}$$

$$\frac{d^2y}{dt^2} = \frac{Th}{\rho\ A}\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dt^2} = a^2\frac{d^2y}{dx^2} \qquad a^2 = \frac{Th}{\rho\ A}$$



$$y_{tt} = a^2 y_{xx} \quad x \in (0, L), t \geq 0$$

$$y(0,t) = y(L,t) = 0 \quad t \geq 0$$

$$y(x,0) = \alpha(x) \quad y_t\,(x,0) = \beta(x) \quad x \in (0, L)$$

$$y = T(t)X(x)$$

$$y_{tt} = T''(t)X(x)$$

$$y_{xx} = T(t)X''(x)$$

$$T''(t)X(x) = a^2\, T(t)X''(x)$$

$$\frac{T''(t)}{T(t)} = a^2\, \frac{X''(x)}{X(x)} = \lambda^2$$

$$\frac{T''(t)}{T(t)} = a^2 \frac{X''(x)}{X(x)} = \lambda^2$$

$$\frac{T''(t)}{T(t)} = -\lambda^2$$

$$a^2 \frac{X''(x)}{X(x)} = -\lambda^2$$

$$a^2\, X''(x) + \lambda^2\, X(x) = 0$$

$$X(x) = \; A\, Sin\left(\frac{\lambda}{a}x\right) + \; B\; Cos\left(\frac{\lambda}{a}x\right)$$

$$T''(t) + \lambda^2 T(t) = 0$$

$$T(t) = \; C\; Sin(\lambda\, t) + \; D\; Cos(\lambda\, t)$$

$$X(x) = A\,Sin\left(\frac{\lambda}{a}x\right) + B\,Cos\left(\frac{\lambda}{a}x\right)$$

$$T(t) = C\,Sin(\lambda\,t) + D\,Cos(\lambda\,t)$$

$$X(0) = 0 \rightarrow B = 0$$

$$X(L) = 0 \rightarrow Sin\left(\frac{\lambda}{a}L\right) = 0 \rightarrow \frac{\lambda}{a}L = n\,\pi$$

$$\lambda = \frac{n\,\pi\,a}{L}$$

$$Xi(x) = Ai\ Sin\left(\frac{n\,\pi}{L}x\right)$$

$$Ti(t) = \ C\ Sin\left(\frac{n\,\pi\ a}{L}\ t\right) + \ D\ Cos\left(\frac{n\,\pi\ a}{L}\ t\right)$$

$$y = X(x)T(t)$$

$$y(x,t) = \sum_{n=1}^{\infty} Sin\left(\frac{n\pi}{L}x\right)\ \left(a_n Cos\frac{n\pi a}{L}\ t + b_n Sin\frac{n\pi a}{L}\ t\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} Sin\left(\frac{n\pi}{L}x\right)\left(a_n Cos\frac{n\pi a}{L}\,t + b_n Sin\frac{n\pi a}{L}\,t\right)$$

$$a_n = \frac{2}{L}\int_0^L \alpha(x)\,Sin\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{2}{a\,n\,\pi}\int_0^L \beta(x)\,Sin\frac{n\pi}{L}x\,dx$$

$$y(x,t) = \sum_{n=1}^{\infty} Sin\left(\frac{n\pi}{L}x\right)\left(a_n Cos\frac{n\pi a}{L}t + b_n Sin\frac{n\pi a}{L}t\right)$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$p_n = \sqrt{{a_n}^2 + {b_n}^2}$$

$$y(x,t) = \sum_{n=1}^{\infty} p_n \sin\frac{n\pi}{L}x \ \ \cos\frac{n\pi}{L}a\,(t-\gamma)$$

$$y(x,t) = \sum_{n=1}^{\infty} Sin\left(\frac{n\pi}{L}x\right)\left(a_n Cos\frac{n\pi a}{L}\ t + b_n Sin\frac{n\pi a}{L}\ t\right)$$

$$\frac{n\pi}{L}a\,\tau = 2\,\pi$$

$$\tau = \frac{2\,L}{n\,a}$$

$$f_n = \frac{1}{\tau} = n\ \frac{a}{2\,L}$$

$$f_n = \frac{1}{\tau} = n\ \frac{a}{2\ L}$$

$$a = \sqrt{\frac{Tensión\ [N]}{densidad\ lineal\ [\frac{kg}{m}]}}$$

$$f_n = \frac{n}{2\ L}\ \sqrt{\frac{Th}{\rho\ A}}$$

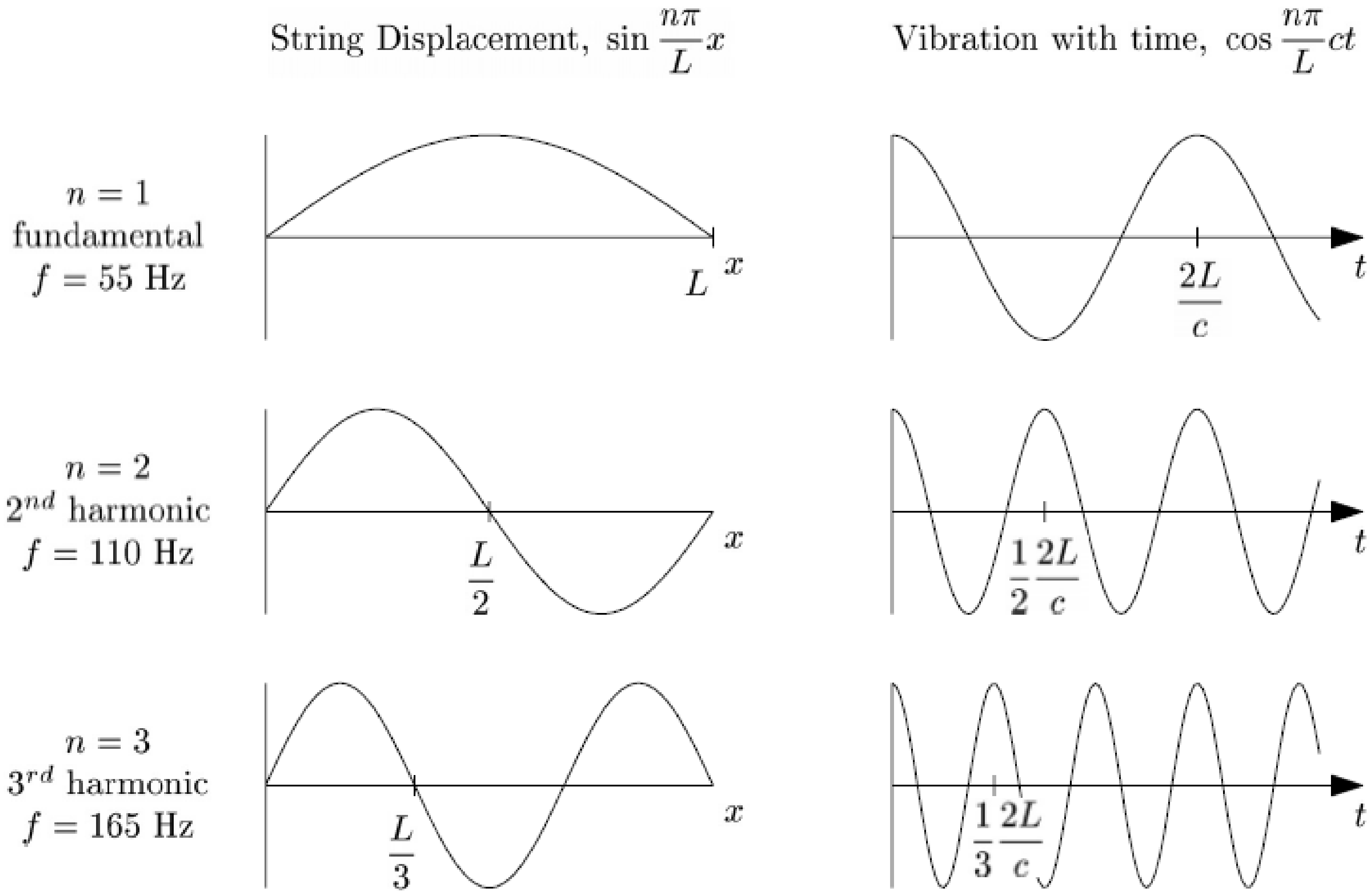
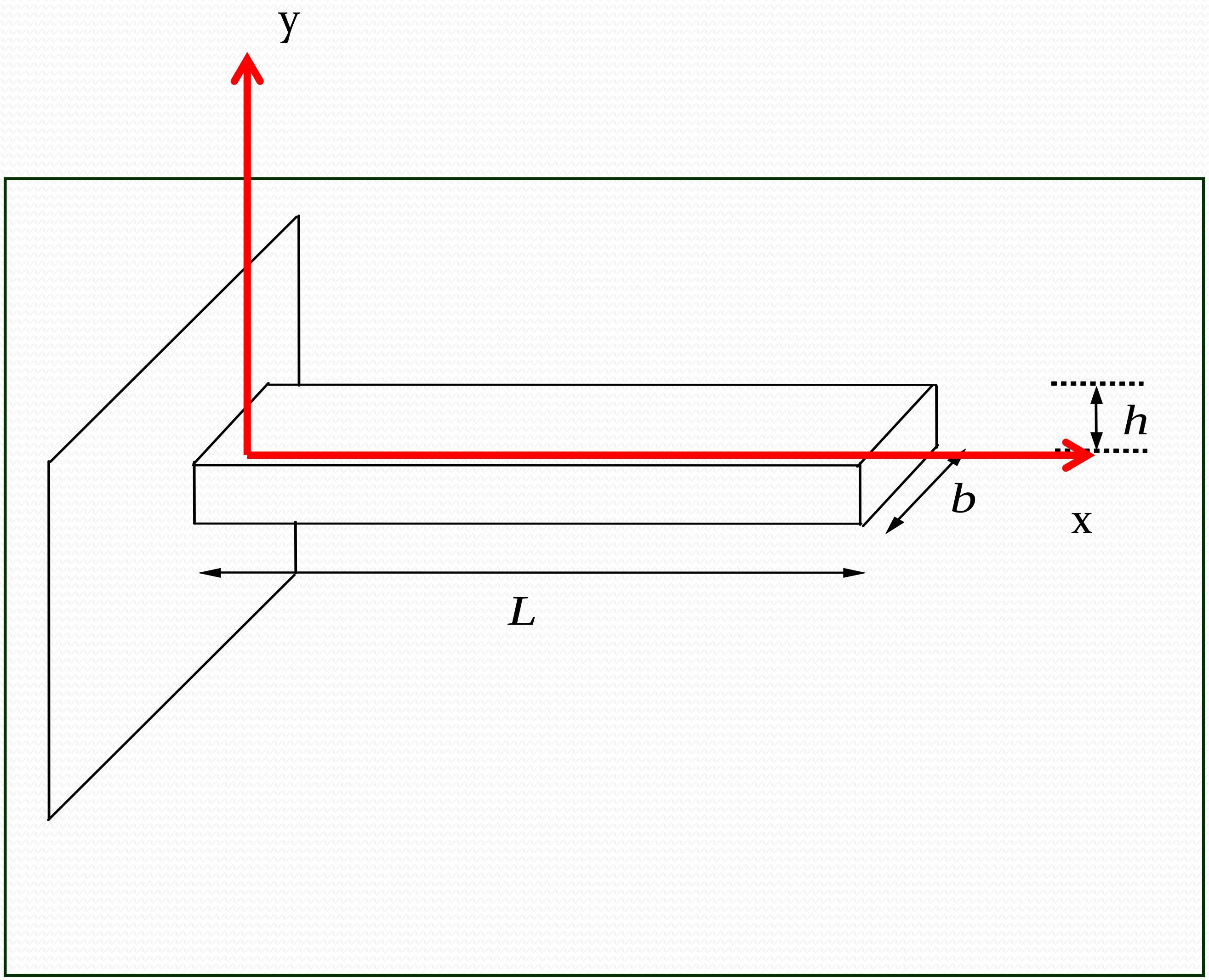


Figure 1. Modes of a vibrating string.

El diapasón y la viga en voladizo.

Una viga en voladizo, de sección constante



$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = f(x)$$

$$EI\frac{d^4y}{dx^4} = f(x)$$

$$M = -EI\frac{d^2y}{dx^2}$$

$$Q = -\frac{d}{dx}EI\frac{d^2y}{dx^2}$$

$$\mathcal{L} = T - V$$

$$\mathcal{L} = \frac{1}{2}\rho A\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}E\,I\left(\frac{\partial^2 y}{\partial x^2}\right)^2 + f(x)\,y(x,t)$$

$$\mathcal{L} = \frac{\rho A}{2}{y_t}^2 - \frac{E\,I}{2}\,{y_{xx}}^2 + f\,y$$

$$\mathcal{L} = \frac{\rho A}{2}\dot{y}^2 - \frac{E\,I}{2}\,{y_{xx}}^2 + f\,y$$

$$\mathcal{L} = \frac{\rho A}{2}\dot{y}^2 - \frac{E\,I}{2}\,{y_{xx}}^2 + f\,y$$

$$\frac{\partial \mathcal{L}}{\partial \dot{y}} = \rho A\,\dot{y} \qquad \frac{\partial \mathcal{L}}{\partial y_{xx}} = -E\,I\,y_{xx} \qquad \frac{\partial \mathcal{L}}{\partial y} = f$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial y_{xx}}\right) = 0$$

$$f - \rho A\,\ddot{y} + (-E\,I\,y_{xx})_{xx} = 0$$

$$f - \rho A\, \ddot{y} + (-E\, I\, y_{xx})_{xx} = 0$$

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = -\rho A\frac{\partial^2 y}{\partial t^2} + f(x)$$

$$EI\frac{\partial^4 y}{\partial x^4} = -\rho A\frac{\partial^2 y}{\partial t^2} + f(x)$$

$$EI\frac{\partial^4 y}{\partial x^4} + \rho A\frac{\partial^2 y}{\partial t^2} = 0$$

$$EI\frac{\partial^4 y}{\partial x^4}+\rho A\frac{\partial^2 y}{\partial t^2}=0$$

$$EI\,F(y)^{(iv)}\,G(t)+\rho A\,F(y)G(t)''=0$$

$$\frac{EI}{\rho A}\frac{F(y)^{(iv)}}{F(y)}=\omega^2=-\,\frac{G(t)''}{G(t)}$$

$$\frac{EI}{\rho A}\,F(y)^{(iv)}-\omega^2F(y)=0$$

$$G(t)''+\omega^2\,G(t)=0$$

$$\frac{EI}{\rho A} F(y)^{(iv)} - \omega^2 F(y) = 0$$

$$F(x) = C1\, cos(\beta\, x) + C2 \sin(\beta\, x) + C3\, cosh\, (\beta\, x) + C4\, sinh(\beta\, x)$$

$$\beta = \left(\frac{\rho A\, \omega^2}{E\, I}\right)^{1/4}$$

$$G(t)'' + \omega^2\, G(t) = 0$$

$$G(t) = C5\, cos(\omega\, t) + C6 \sin(\omega\, t)$$

$$Y(x,t) = F(x)G(t)$$

$$G(t) = C5\,cos(\omega\, t) + C6\,\sin(\omega\, t)$$

$$F_n(x)$$
$$= C1\,cos(\beta_n\, x) + C2\,\sin(\beta_n\;\, x)$$
$$+\;\, C3\,cosh(\beta_n\;\, x) + C4\,sinh(\beta_n\;\, x)$$

$$\beta = \left(\frac{\rho A\, {\omega_n}^2}{E\, I}\right)^{1/4}$$

Condiciones de Frontera.

$$Si\ x = 0, \qquad F_n(0) = 0, \qquad F'(0) = 0$$

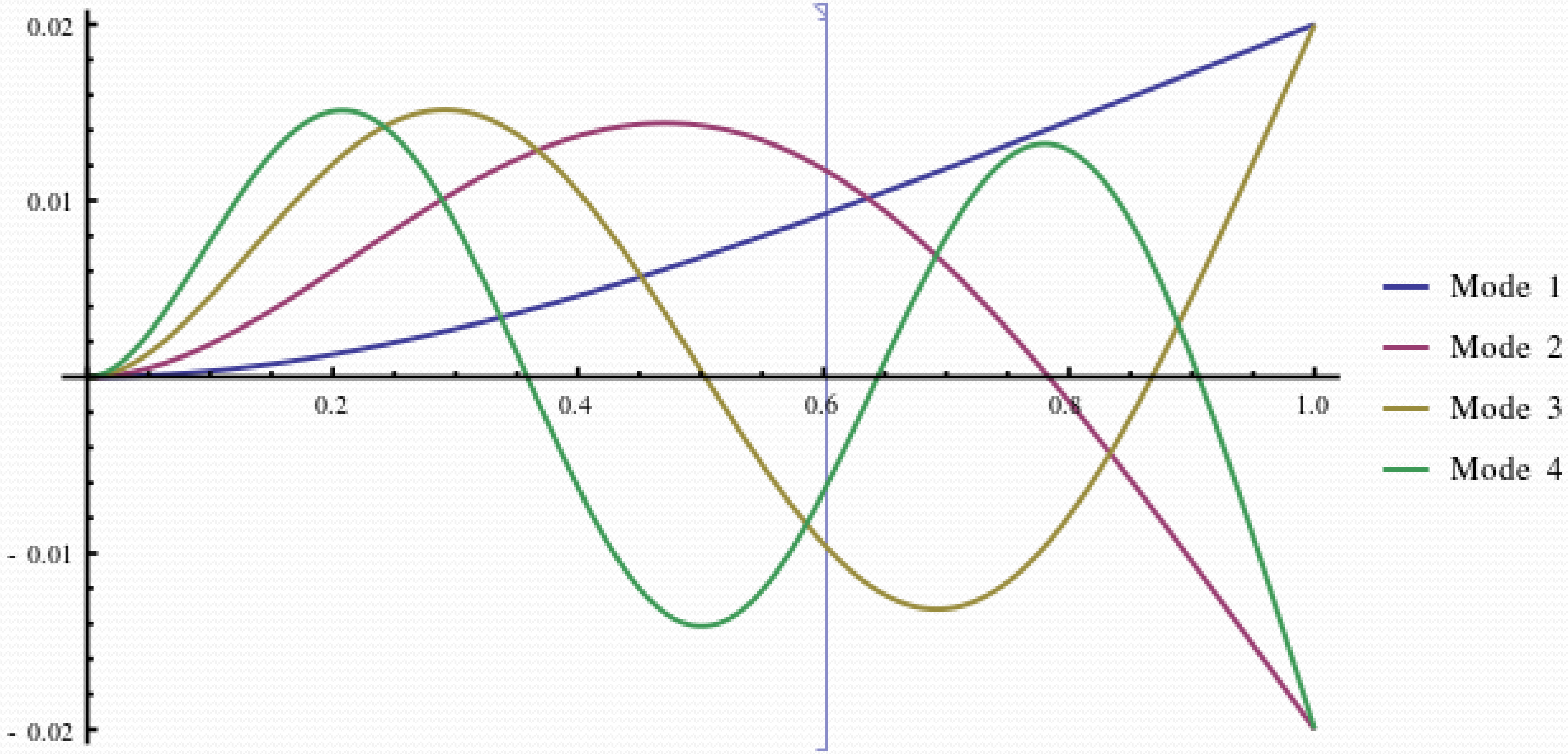
$$Si\ x = L, \qquad F''_n(L) = 0, \qquad F'''(0) = 0$$

$$cosh(\beta_n\, x)\, cos(\beta_n\, x) + 1 = 0$$

$$F_n(x) = C1\, cos(\beta_n\, x) + C2 \sin(\beta_n\ \ x)$$

http://en.wikipedia.org/wiki/Euler%E2%80%93Bernoulli_beam_theory#mediaviewer/File:Beam_mode_6.gif

Modos de vibración



$$\omega_n = 11.0\sqrt{\frac{E\ I}{\mu\ L}}$$

$$\omega_n = Frecuencia\ natural\ fundamental\left[\frac{rad}{seg^2}\right]$$

$$E = Módulo\ de\ Young\ del\ material\ [Psi]$$

$$I = Momento\ de\ inercia\ de\ la\ sección\ transversal\ \ [in^4]$$

$$\mu = masa\ por\ unidad\ de\ longitud\ \left[\frac{lbm}{in}\right]$$

$$L = Longitud\ de\ la\ viga\ [in]$$

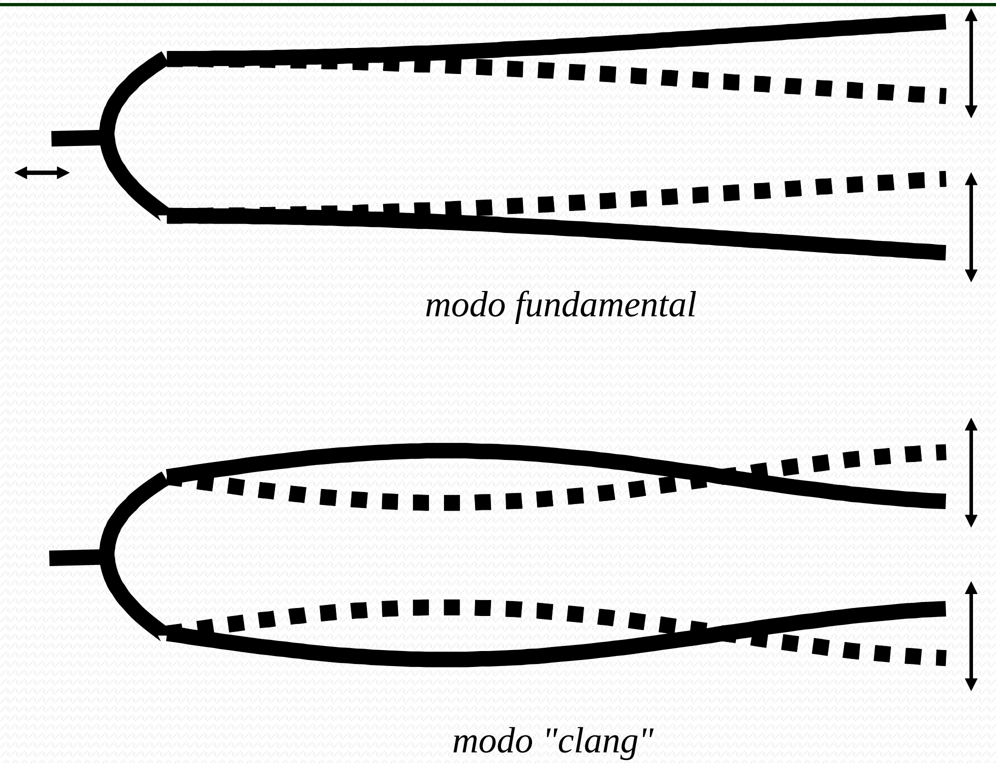
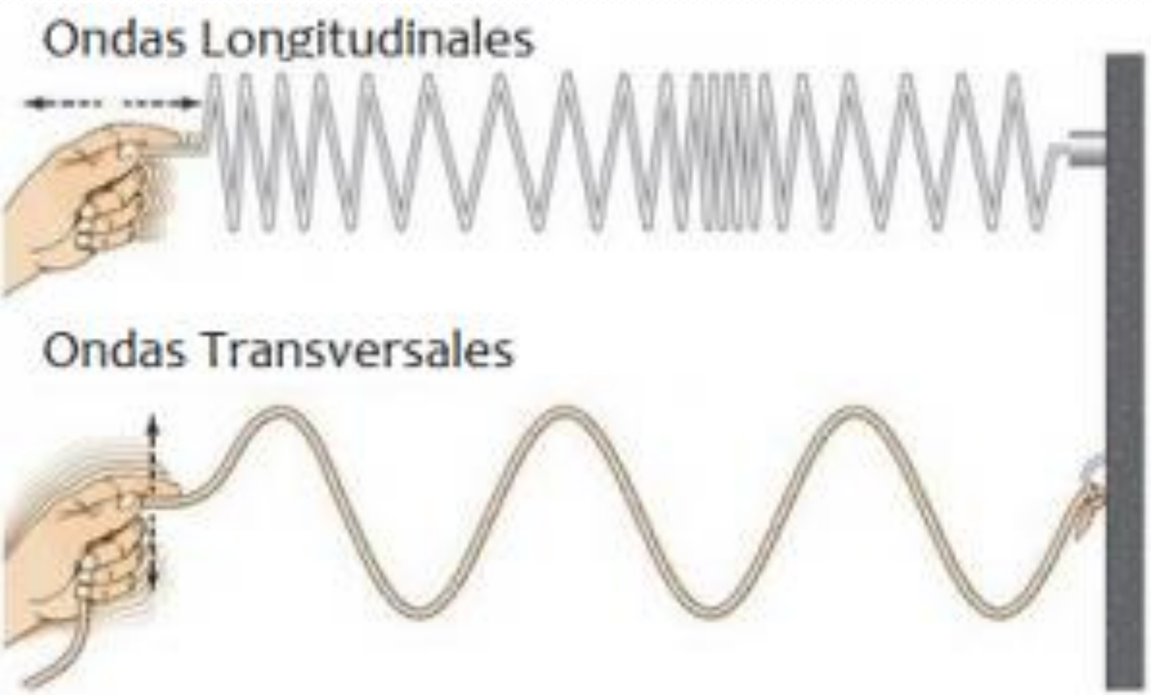


Figura 7. El modo fundamental y el modo “clang” de un diapasón. La frecuencia del modo clang es aproximadamente 6 veces la frecuencia del modo fundamental.

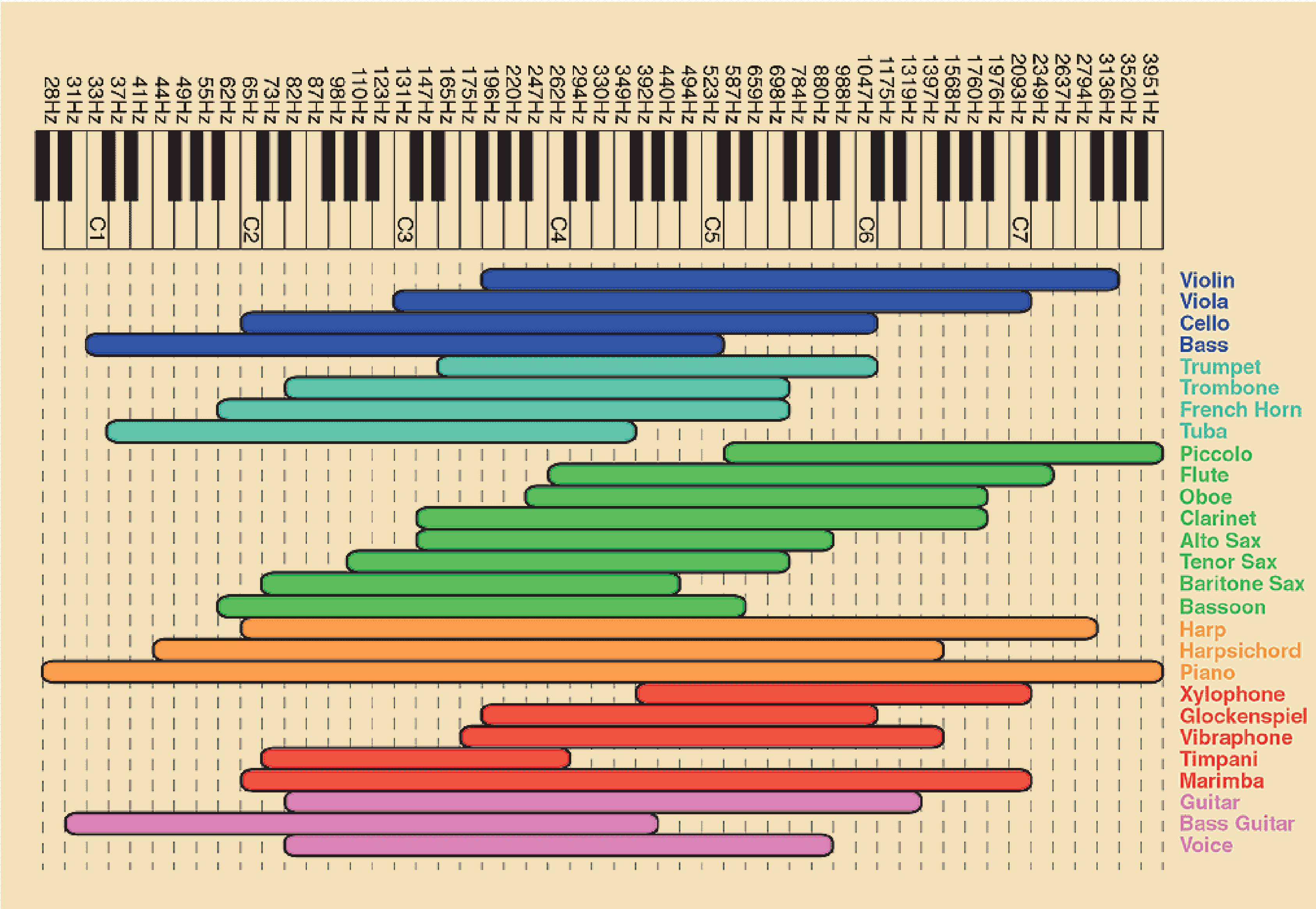
http://en.wikipedia.org/wiki/Euler%E2%80%93Bernoulli_beam_theory#mediaviewer/File:Beam_mode_6.gif


Ondas Longitudinales
Ondas Transversales

Velocidad del sonido.

- En la atmosfera terrestre es de 343 [m/s] (a 20 [°C] de temperatura, con 50% de humedad y a nivel del mar.
- En el agua (a 25 °C) es de 1493 m/s.
- En la madera es de 3700 m/s.
- En el hormigón es de 4000 m/s.
- En el acero es de 6100 m/s.
- En el aluminio es de 6400 m/s.

Frecuencias de las notas musicales.


28Hz
31Hz
33Hz
37Hz
41Hz
44Hz
49Hz
55Hz
62Hz
65Hz
73Hz
82Hz
87Hz
98Hz
110Hz
123Hz
131Hz
147Hz
165Hz
175Hz
196Hz
220Hz
247Hz
262Hz
294Hz
330Hz
349Hz
392Hz
440Hz
494Hz
523Hz
587Hz
659Hz
698Hz
784Hz
880Hz
988Hz
1047Hz
1175Hz
1319Hz
1397Hz
1568Hz
1760Hz
1976Hz
2093Hz
2349Hz
2637Hz
2794Hz
3136Hz
3520Hz
3951Hz
C1
C2
C3
C4
C5
C6
C7
Violin
Viola
Cello
Bass
Trumpet
Trombone
French Horn
Tuba
Piccolo
Flute
Oboe
Clarinet
Alto Sax
Tenor Sax
Baritone Sax
Bassoon
Harp
Harpsichord
Piano
Xylophone
Glockenspiel
Vibraphone
Timpani
Marimba
Guitar
Bass Guitar
Voice

Nota	Frecuencia [Hz]
La	440
La # - Si *b*	$440 \times 2^{1/12}$
Si	$440 \times 2^{2/12}$
Do	$440 \times 2^{3/12}$
Do # - Re *b*	$440 \times 2^{4/12}$
Re	$440 \times 2^{5/12}$
Re # - Mi *b*	$440 \times 2^{6/12}$
Mi	$440 \times 2^{7/12}$
Fa	$440 \times 2^{8/12}$
Fa # - Sol *b*	$440 \times 2^{9/12}$
Sol	$440 \times 2^{10/12}$
Sol # - La *b*	$440 \times 2^{11/12}$
La	$440 \times 2^{12/12} = 880$

https://www.youtube.com/watch?v=iIQFb9PRbmk

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